Sovereign Debt Auctions with Strategic Interactions *

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Abstract

In this paper, we study the impact that alternative ways of issuing sovereign debt have on borrowing decisions, the cost of debt, and welfare. We build a model of sovereign borrowing and default with repeated auctions, disciplined with proprietary bid level data. We calibrate the model to the Portuguese economy and use it to perform a counterfactual, comparing the two most common types of auction: uniform and discriminatory price auctions. We find that switching to a uniform protocol constitutes a Pareto improvement, and that the difference in welfare is highest during crises (up to 0.6% of permanent consumption in the small open economy). This result aligns with the observed switch to a uniform protocol in Portugal following the sovereign debt crisis of the 2010s. We find that accounting for dynamic effects is crucial. In a single auction setting, given standard values for risk aversion of the government, the discriminatory protocol is optimal. However, with repeated auctions, the insurance properties of the discriminatory protocol lead to over-borrowing. This mechanism, and its effect on prices, makes the uniform protocol a better option. Finally, the calibrated model generates spreads with a volatility that significantly exceeds their mean, as in the data, a documented shortcoming of previous sovereign debt models.

JEL Codes: D44, E43, F34, F41, G15, H63.

Keywords: Sovereign debt auctions, default risk, discretion, dilution.

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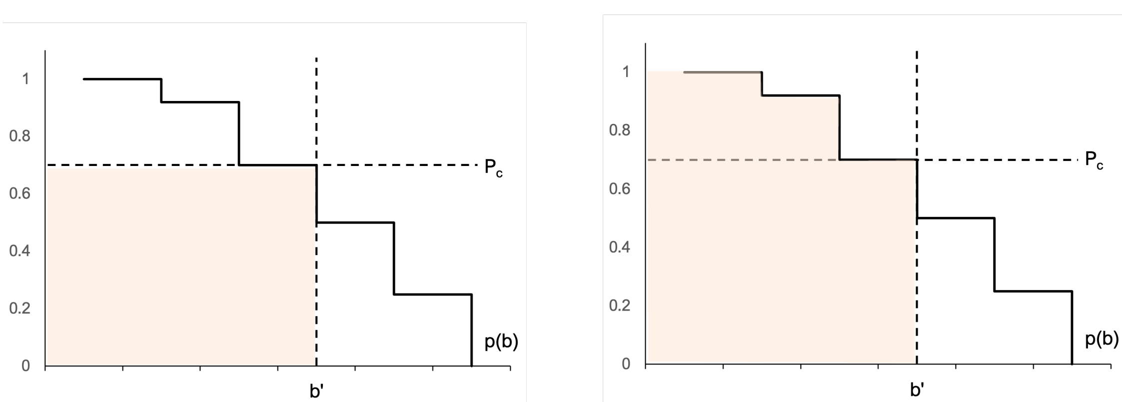
1 Introduction

Governments of both Emerging Market Economies (EMEs) and Advanced Economies (AEs) maintain enormous stocks of sovereign debt.[1] Most of this debt is issued in auctions. Because of the massive amounts involved, both policymakers and academics have been extremely interested in determining the best way to run these auctions. As stated in Chari and Weber (1992) "with such large amounts at stake, even small improvements in the Treasury's auction procedure can lead to large gains for taxpayers." Indeed, this interest dates back at least to Friedman (1960). In testimony to the Joint Economic Committee in 1959, Milton Friedman asserted: "The present method [to issue debt] involves payment of different prices by different purchasers [...]. A preferable alternative is to ask purchasers to specify the amounts they are willing to buy at a schedule of prices, determine a price so as to clear the market, and charge all purchasers that single price".

In sovereign debt auctions, investors submit bids consisting of the highest price they are willing to pay to purchase a unit of debt, and how much they are willing to buy. Then, the government chooses which bids to accept. There is wide variation across countries in auction protocols, i.e. the set of rules determining how much each winning bid pays. OECD (2023) found 40 of 41 countries surveyed used auctions. Of those, 12 used uniform price auctions, 15 used discriminatory price auctions and 13 used both.



(a) A uniform price auction

(b) A discriminatory, or "pay-as-bid," auction

Figure 1: Comparison of uniform price and discriminatory price auctions

[1]In 2023, AEs total government debt to GDP was over 110% (EMEs stood at nearly 70%).

Figure 1 depicts how these two protocols work. Individual bids are combined into an aggregate demand function, $p(b)$. The government selects the amount issued, b', and the clearing price, P_c. In a uniform price auction, all accepted bids are executed at the marginal price. In a discriminatory price auction (pay-as-bid), all accepted bids are executed at their bidding prices. Revenue is depicted by the shaded area below the aggregate demand function. Which auction protocol yields more revenue is not obvious since, as we will show, the aggregate demand function itself depends on the auction protocol.

There are two reasons for this. First, the government has discretion over the quantity sold in each auction, and it chooses that quantity strategically after observing investors' bids. The incentives to issue more or less debt differ with the auction protocol used, so different auction protocols lead to different issuance choices. Those different issuance choices in turn lead to different lender expectations about how much the debt will be worth, which leads to different bidding behavior. Second, there is a dynamic link between debt auctions across time. The value of debt today depends not only on how much debt is issued today, but also on how much debt will be issued next period, and the period after, and so forth. Therefore, the auction protocol can also affect the price of debt today through its effect on the incentives to borrow in the future.

In this paper we ask: How do outcomes (yields, borrowing and default decisions and welfare) depend on the auction protocol used? How do those differences inform what auction protocol countries facing default risk should use?

To answer these questions, we build a theory of how strategic interactions between the borrower and lenders affect auction outcomes, and evaluate how different auction protocols interact with default risk. We fill a gap in the literature of sovereign debt and default: the role played by how debt is issued and how primary market prices are determined. At the same time, we also contribute to the literature that studies differences in these two types of auctions, which has been centered on environments with exogenous supply and a single auction. While our application focuses on sovereign debt, our conclusions apply to other settings where a seller reserves discretion over quantity and default risk is a concern. In particular, they may also apply to auctions of corporate debt.

We begin by studying a two period environment with a single auction and default risk. The foreign investors buying the debt are symmetric and competitive. We study the two protocols. In this setting, we first show that if the debt issuance policy of the government is identical across protocols then revenue equivalence arises: both auction protocols generate the same expected revenue. However, even though ex-ante revenue is equal across auction protocols, we find that the bid functions are different. Investors bid lower prices under a discriminatory price protocol than under a uniform price protocol. Under a uniform price protocol, all winning bids are executed at the marginal price, so competition results in marginal prices exactly matching the unit value of the debt issued. On the other hand, when investors pay-as-bid, we see the classic "winner's curse." Investors fear their bid will not be marginal, that the government may issue more debt, and they may end up paying a high price for a low value asset. As a result, they bid lower prices. However, under a discriminatory price protocol, the average executed bid is above the marginal price of the auction. As the government is faced with different financing needs, the average executed bid has a lower variance than the marginal price of the auction. This contrasts with the uniform price auction, where the average price paid by investors is exactly the marginal price of the auction. When revenue equivalence holds, a risk neutral government is thus indifferent between protocols, but a risk averse government prefers the discriminatory price protocol because the variance of the average executed price, and that of revenue, is smaller.

When debt is chosen strategically, revenue equivalence need not hold. The difference in auction protocol creates different incentives for the government to borrow, which leads to different issuance policies, violating the sufficient condition, of identical debt issuance policies, that guarantees revenue equivalence. Even in a static environment, the choice of auction protocol has an impact on expected revenue, yields, borrowing, and welfare. The ranking of auction protocols depends on the government's preferences, particularly on how much it values smoothing revenue. When government preferences are linear, the effects of static dilution are extreme and the government prefers the uniform price protocol. Convex preferences create a motive to smooth consumption over time, which disciplines the government's borrowing, reducing the effects of static dilution. As a result, for sufficiently concave utility, the government prefers the discriminatory price protocol. This is related to the trade-off between levels and variance of prices identified above, reminiscent of Cole et al. (2018).[2] Having illustrated how incentives to borrow depend on the auction protocol in a two period setting with a single auction, we then move to an infinite horizon setting where future incentives to borrow are affected by each protocol.

We extend a standard framework for studying government borrowing and default to allow for different auction protocols[3]. We inform our modeling decisions using proprietary bid level data for Portuguese sovereign debt auctions (as first used in Alves Monteiro (2022)). We observe that: (i) individual bid functions tend to be homogeneous in normal times; (ii) during the crisis period (2008-2014) the aggregate bid function becomes steeper and more inelastic; and, (iii) there is no evidence of persistent differences between investors. Since higher than expected government spending played a key role in the Eurozone Debt Crises (see Copelovitch et al. (2016)), we incorporate uncertainty about required government expenditures as the primary source of uncertainty regarding the government's need/desire to borrow.

In the standard sovereign debt model, long term debt creates dynamic dilution motives that make the equilibrium allocation constrained inefficient.[4] Essentially, when the government inherits legacy debt, it has an incentive to issue new claims on the resources it has already "earmarked" to pay its legacy investors. This leads the government to borrow more than it would have planned to ex-ante, and lenders react to this by offering it lower prices in anticipation. We show that different auction protocols result in differently shaped revenue curves for the government. Under the uniform price protocol, declines in the marginal price apply to all debt issued, while under the discriminatory price protocol,

[2]Their result arises from "sufficiently asymmetric information" between investors. Ours rely on debt being strategically chosen by the government. In our setting, when utility is concave enough, the discriminatory price protocol allows the government to achieve significantly lower variance in prices without the massive drop in the mean price that occurs under linear utility.

[3]Refer to the literature review in section 2 for an exposition of the standard framework used for studying sovereign borrowing and default.

[4]Specifically, if the government could commit ex-ante to a sequence of debt issuance choices while still lacking commitment with respect to default, it could achieve a strictly better outcome. See Aguiar and Amador (2019) for a full, technical treatment of this result, as well as a proof that the allocation with short term debt is constrained efficient.

they only apply to the marginal unit issued. This supercharges the dilution motives that arise under the discriminatory price protocol.

In a quantitative exercise, we discipline the model using the experience of Portugal until 2011 (when it was bailed out by the European Commission and Central Bank and the International Monetary Fund). During this period, Portugal used the discriminatory price protocol for all auctions. The calibrated model is able of matching standard moments in the Portuguese economy regarding debt, spreads and business cycles statistics. Moreover, the use of a discriminatory protocol lets the model easily generate spreads whose volatility significantly exceeds their mean, a shortcoming of previous sovereign debt models, as documented in Aguiar et al. (2016). In a counterfactual, we compare the two auction protocols. We find that the uniform price protocol yields higher welfare than the discriminatory price protocol, and that these gains are highest during crises. Moreover, switching to a uniform price protocol constitutes a Pareto improvement as both the small open economy and foreign investors are better off after the switch. This result is consistent with the observed switch in 2011 to a uniform price protocol for long term debt.[5]

Accounting for dynamic effects is crucial for this welfare result. Given standard values for risk aversion of the borrowing country, we would find in the static setting with a single auction that the discriminatory price protocol is optimal. The dynamic effects of the discriminatory price protocol, however, are terrible. The uniform price protocol provides much better incentives for borrowing over time, as well as protecting investors from static dilution within an auction. These both lead to much better bond prices for the government. In fact, under the calibrated model, these gains more than justify forgoing the insurance mechanism provided by the discriminatory protocol.[6]

[5] In the midst of the crisis, Portugal stopped issuing securities with maturity longer than one year from 2011 to 2014. When the Portuguese Treasury resumed auctioning debt at those maturities in 2014, it used the new protocol.

[6] While the fact that the gains from insurance are relatively small is consistent with the well known fact that the welfare costs of fluctuations are small, as in Lucas (1987), the existing literature is silent as to the size of the gains from reducing each kind of dilution.

2 Literature Review

This paper builds on the sovereign debt literature, with an emphasis on the auction framework used to issue debt. Related papers here include Cole et al. (2018), Pycia and Woodward (2023) and Cole et al. (2022). Each aims at comparing the two auction protocols. To do so, each considers a static auction model with asymmetric information across bidders and exogenous asset quality. Cole et al. (2022) in particular, identifies the insurance mechanism that we also describe for the discriminatory price protocol. Apart from Pycia and Woodward (2023), all consider exogenously random supply of debt. In Pycia and Woodward (2023), the government commits to a distribution for the supply and a reserve price before observing demand. We focus instead on incorporating different auction protocols into an infinite horizon, dynamic model of government borrowing and default. This paper is the first to consider a strategic government that has discretion over supply and can choose how much to issue after observing demand. We show that this strategic interaction between a government with discretion and optimizing investors matters. In particular, investors know that distinct protocols induce different debt issuances by the government (which may break revenue equivalence between protocols).

This paper also builds on the quantitative sovereign default literature, which is based on the classic setting of Eaton and Gersovitz (1981). Key early papers include Aguiar and Gopinath (2006), Arellano (2008), Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). One insight of the later papers is that incorporating long term debt is crucial for being able to match the levels of debt and levels and volatility of interest rate spreads observed in Emerging Market Economies. Many branches of the literature build on this workhorse model with long term debt[7]. We also build on this setting by explicitly modeling the auction protocols countries use to issue debt. We are then able to assess the role played by how debt is issued and how primary market prices are determined. We find that explicitly modeling the auction framework enriches the environment, leading to two interesting phenomena. First, the use of a discriminatory price protocol lets

[7] Conesa and Kehoe (2017) and Bocola and Dovis (2019) focus on the role of rollover risk and self-fulfilling crises. Arellano and Ramanarayanan (2012), Sánchez et al. (2018), Bocola and Dovis (2019) and Dvorkin et al. (2021) focus on the role of maturity choice.

the model easily generate spreads whose volatility significantly exceeds their mean (a key feature of the Eurozone countries that went through debt crises in 2008-2014, and a notable difference of those countries from the EMEs the sovereign default literature had previously focused on). Previous models (see Aguiar et al. (2016)) could not generate this pattern without producing counterfactual levels of debt or spreads[8]. Second, we find that discriminatory price protocols are prone to self-fulfilling crises even in environments where such crises would not be possible under a uniform price protocol. We leave the discussion of this second phenomenon to a companion paper Alves Monteiro and Fourakis (2023).

There is a large auction theory literature that studies multi-unit auctions. In these auctions, bidders submit both prices and quantities, generating a two dimensional strategic problem[9]. We assume that investors are infinitesimal (and therefore take aggregate bid function as given). This allows us to focus on how the auction protocols determine prices by aggregating bids, while avoiding, similarly to Cole et al. (2018), strategic considerations between investors. Instead, we focus on the strategic interaction that arises from having a maximizing government that chooses how much debt to issue after observing bids, reserving discretion on the quantity sold. Both Back and Zender (2001) and McAdams (2007) allow for discretion on the quantity sold. Their use of discretion, however, is orthogonal to ours. They use the possibility of discretion as a tool to make a theoretical point on how to avoid collusive equilibria in uniform price auctions, in a static environment without default risk.

There is previous work on how multi-unit auctions determine prices in equilibrium, particularly from an empirical perspective. Hortaçsu (2002) presents a model based on Wilson (1979) of a multi-unit discriminatory-price auction with a finite number of symmetric

[8]See for instance Bocola et al. (2019). Paluszynski (2023) does get closer, however, in the long run simulations, still generates counterfactual levels of debt and spreads.

[9]See Wilson (1979) for an early paper on multi-unit auctions, Engelbrecht-Wiggans and Kahn (1998) for a multi-unit auction where investors are allowed to buy more than one unit of the good. Perry and Reny (1999) show that the linkage principle – that revenue increases as more information about the quality of the good is provided to bidders – does not hold in general for multi-unit auctions. McAdams (2006) shows existence of a monotone pure strategy equilibrium when bidders are risk neutral with independent multi-dimensional types and interdependent values.

risk-neutral bidders with independent private values. In their model a single bid affects the bid functions through changes in the distribution of the price that clears the auction. They construct a non-parametric estimator of the distribution exploiting a re-sampling technique. Kastl (2011) builds on this framework by allowing for discrete-step bid functions. Our price-taking assumption allows us to abstract from this problem. Our auction framework also differs in that we assume a common valuation for the debt being auctioned, instead of the independent private values assumption in the cited work. That is, the value of debt is pinned down by the future endogenous probability of default and investors know, for each possible debt issuance, the value of debt in equilibrium.

Since Milton Friedman's suggestion that the government should issue debt through uniform price auctions, there have been several empirical studies that compare the outcome under the two protocols. Hortaçsu (2002) and Barbosa et al. (2022) find no statistically significant differences in revenues, Février et al. (2002), Kang and Puller (2008), Armantier and Lafhel (2009), Marszalec (2017), Hattori and Takahashi (2022), and Mariño and Marszalec (2023) find slightly higher revenues in discriminatory price, and Castellanos and Oviedo (2008), Armantier and SbaÏ (2006), and Armantier and Sbaï (2009) find slightly higher revenues in uniform price. This shows how unsettled the debate on whether discriminatory or uniform price auctions raise higher expected revenues for countries facing default risk.

Theoretical results comparing the two protocols also vary, particularly as different environments and assumptions are considered. Swinkels (2001) used an environment where variance in valuation across investors dissipates as the number of investors grows large (asymptotic environmental similarity assumption) and showed that pay-as-bid and uniform price are revenue and welfare equivalent. The same result is obtained by Jackson and Kremer (2006) under the assumption that the proportion of supply to the number of bidders goes to zero. Wang and Zender (2002) considers a model with symmetric investors and common value for the good being auctioned and find pay-as-bid revenue superior when bidders are risk neutral, due to multiplicity and bidder collusion under the uniform price specification. Finally, Pycia and Woodward (2023) finds that under assumptions commonly imposed in empirical work, the two formats are revenue and welfare equivalent. In this paper, we highlight the importance of considering a strategic government with discretion over issuance decisions in a dynamic environment. With this, we are able to assess how the protocols compare taking into account how they affect government and investor decisions over time, instead of comparing protocols for a given fixed, debt issuance policy.

3 Data: Background and Evidence

Auction data was provided by the *Portuguese Treasury and Debt Management Agency* (IGCP, Portuguese acronym). The data comprises all auctions of treasury bills (short maturities) and bonds (long maturities) held from 2003 and 2004, respectively, and up to 2020. Most importantly, the data comprises all individual bids (price and amount) that were placed in each auction, even if they were not executed.

Issuance of treasury bills in the primary market is done through auctions. Treasury bonds are launched for the first time in syndicated operations[10]. New issuances of a line that has already been launched are done through auctions. The IGCP uses a primary dealership model to issue bills and bonds. Only primary dealers, a group of financial intermediaries, participate in the auctions. Dealers are permitted to submit multiple bids as long as the total value does not exceed the upper limit of the overall amount announced for the auction.

Table 1 presents some summary data for the most common bill and bond auctions. One can observe 400 Treasury bill auctions and 161 Treasury bond auctions. The most common maturities are 12 and 3 months for the treasury bills and 10 and 5 years for the treasury bonds. In bill auctions the number of bids averages 39 and in bond auctions it averages 56. Dealers (mean) refer to the average number of dealers present in the auctions of each security. Steps (mean) refer to the average number of bids submitted by a

[10] A syndicate is a group of banks that is given the mandate to place a specific amount of government bonds. It follows a book building process that allows for permanently monitoring of orders and intervention in the allocation of such orders by the IGCP.

dealer. Issued (mean, M€) refer to the average amount issued by the IGCP in auctions of each security.

Table 1: Summary Data on Treasury Bond and Bill auctions

Maturity	Auctions	Bids (mean)	Dealers (mean)	Steps (mean)	Issued (mean, M€)
3 Months	101	35.2	14.5	2.4	471.0
6 Months	88	36.4	14.7	2.4	505.6
12 Months	101	44.0	15.4	2.8	1,037.5
All Bills	400	38.7	14.8	2.5	703.1
5 Years	21	55.9	18.9	2.8	732.3
6 Years	14	56.5	18.2	3.0	754.1
10 Years	52	59.1	17.9	3.2	805.8
All Bonds	161	56.4	17.9	3.0	756.0

Other data sets are further detailed in section 6 where we calibrate the model to the Portuguese economy.

In the next subsections we provide evidence on key aspects that will be used to discipline the structural model:

1. Debt agencies lack commitment to the target amount that is announced prior to the auction – there is supply uncertainty at the moment of the auction;
2. Leading up and during the crisis, investors' bids get more disperse;
3. There is no evidence of persistent heterogeneity across investors;
4. Starting in 2008, public spending was higher than anticipated and there were lower resources to finance it.

3.1 Lack of Commitment and Uncertainty

Lack of Commitment. The week prior to an auction, the IGCP announces the securities being issued and provides a target for the amount it expects to issue. Importantly, there is no commitment to that target. In the data we observe several instances of ex-post deviation from the target. It follows that, although there is a targeted amount, the amount issued in a given auction is uncertain from the bidder's perspective.

Figure 2 highlights this lack of commitment by presenting instances of ex-post deviations from the target in Portuguese auctions. The dots on top of 1 represent auctions in which the amount issued equals the target, while the filled squares represent deviations above or below target. One can see, that even before the debt crisis, the agency would regularly deviate from the ex-ante target.

Brenner et al. (2009) surveyed treasury ministries and central banks around the world and received answers from 48 countries. One of the questions asked was "*Does the treasury (or the central bank) have the right to change the quantity of the debt that is being sold after viewing the demand?*". More than half of the countries that answered (30 out of 48) have some discretion on how much to issue, regardless of a target being announced. This is, once again, evidence of lack of commitment to an ex-ante announcement.

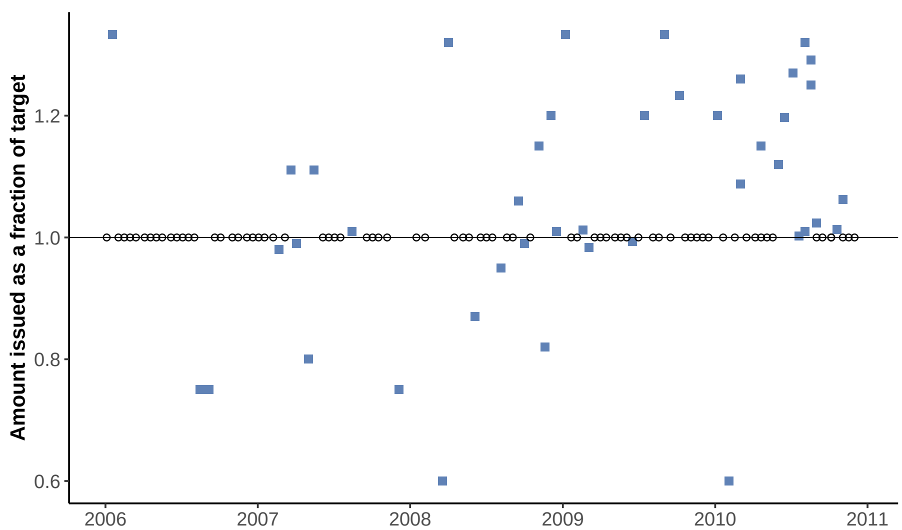


Figure 2: Amount raised as a fraction of the target in Bill Auctions

Debt crisis and uncertainty. During the European debt crisis, debt management offices were purposefully increasing the flexibility of the mechanisms used to issue debt. In April 2010, the IGCP increased this flexibility by: 1) running multiple auctions simultaneously for treasury bonds; 2) providing an interval, instead of an amount, as a target; 3) setting the target range for the sum raised across the auctions being ran simultaneously. It is important to note that April 2010 was right when the sovereign debt crisis in Greece was intensifying, with multiple downgrades of Greek debt and, ultimately, a bailout in May for the Greek government. In February 2011, the same type of changes, just described for treasury bonds, were also introduced in auctions of treasury bills.

In the answers submitted to the 2011 Survey of the OECD Working Party on Public Debt Management, Portugal's debt management office stated: "*In the aftermath of the sovereign debt crisis, the Republic of Portugal (RoP) has resorted to* ***more flexible issuance methods****. Main changes were a more flexible auction calendar and the* ***option of auctioning two bonds simultaneously****.*" (emphasis added).

In that same report, we see that this was not an isolated case: "*In response to uncertainty and volatility, auction calendars have become more flexible in most jurisdictions, auctions were held more frequently and multiple series per auction were introduced.*"

The fact that the debt issuance mechanism is more flexible implies that the use of discretion is even clearer than before the crisis. By providing a range of amounts as target, the government is not committing to any particular issuance amount. This added flexibility can be thought of as a way to ensure that the target (range) is met, i.e. there would be no failed auctions.

3.2 Changes in Demand

To understand uncertainty, we look at bid level data of Portuguese debt auctions. Figure 3 below, highlights changes in demand during the crisis period, as first documented in Alves Monteiro (2022). The figure presents the aggregate bid function for two auctions of one year treasury bills, one in each panel, together with the amount chosen by the IGCP. The bid prices are normalized such that the clearing price is equal to 1. The left panel is representative of demand schedules in normal times whether the panel on the right is representative of demand schedules during the crisis period[11]. We see that, during the crisis, the demand schedule is much steeper and inelastic, with more dispersion of bids. This figure helps understand what is needed to separate the outcomes under the two protocols. If the prices are flat and close to ”risk-free”, as in the left panel, discretion of the government alone is not enough to break equivalence. Not knowing how much the government is going to borrow only matters when different borrowing decisions impact the value of debt, as in the right panel. That is, discretion on the quantity sold together with default risk are the key characteristics of these auctions that separate the outcomes under the two protocols.

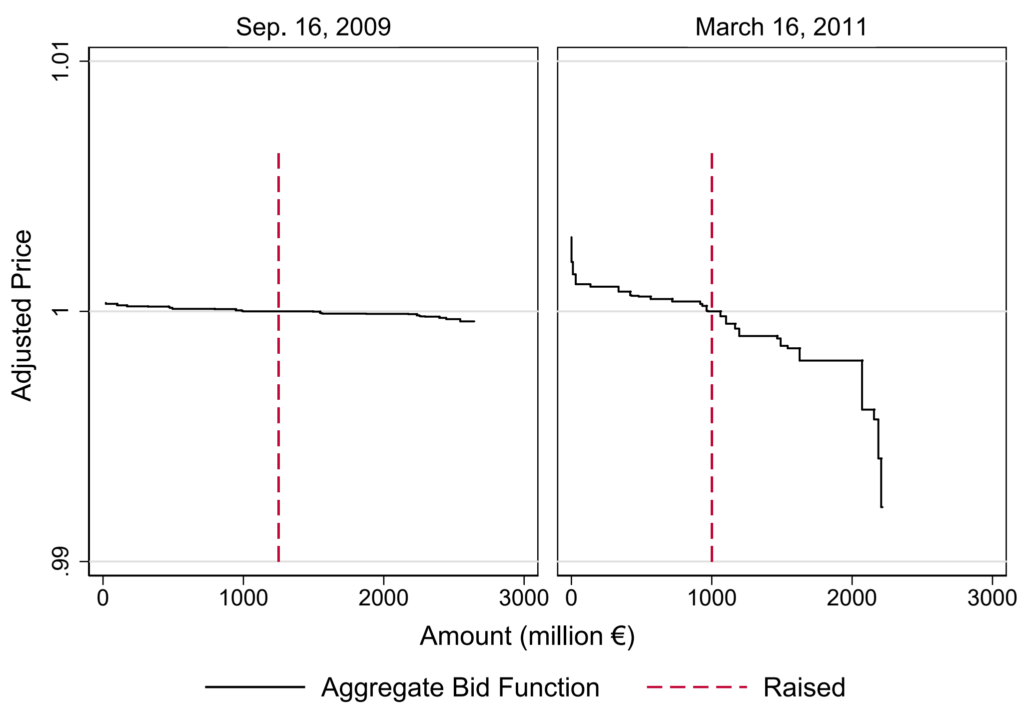


Figure 3: Demand before and during the crisis

The changes depicted in Figure 3 may be driven by differences across dealers or by a greater dispersion within bid functions – all dealers bid at a wider range of prices. Variance across dealers could be indicative of heterogeneity. Variance within bid functions could be explained by greater uncertainty regarding financing needs, borrowing and ulti-

[11]For a thorough analysis of the changes in the demand schedule refer to Alves Monteiro (2022).

mately the value of debt, not necessarily steaming from differences between dealers.

To assess differences across dealers we first look at the variation in the price of the first bid, with the lowest yield (highest price). The first bid in a dealer's bid function is the most likely to be executed as it has the lowest yield. As such, we argue that this bid is also the most informative regarding dealer's characteristics.

Figure 4 presents the standard deviation of the lowest bid yield across dealers, $\{1, \ldots, N\}$, at a given auction, for treasury bills and treasury bonds. More precisely, each point represents the average of such standard deviations across auctions, $\{1, \ldots, M_t\}$, for a given year, t as follows:

$$\overline{SD}_t = \frac{1}{M_t} \sum_{j=1}^{M_t} \sqrt{\sum_{i=1}^{N} \frac{(p_{i1j} - \overline{p}_{1j})^2}{N}}$$

We separate this analysis for bills and bonds as the set of dealers participating in bills and bond auctions are potentially different. Moreover, for bonds, we see a change in protocol during the crisis as well as a hiatus on issuances.

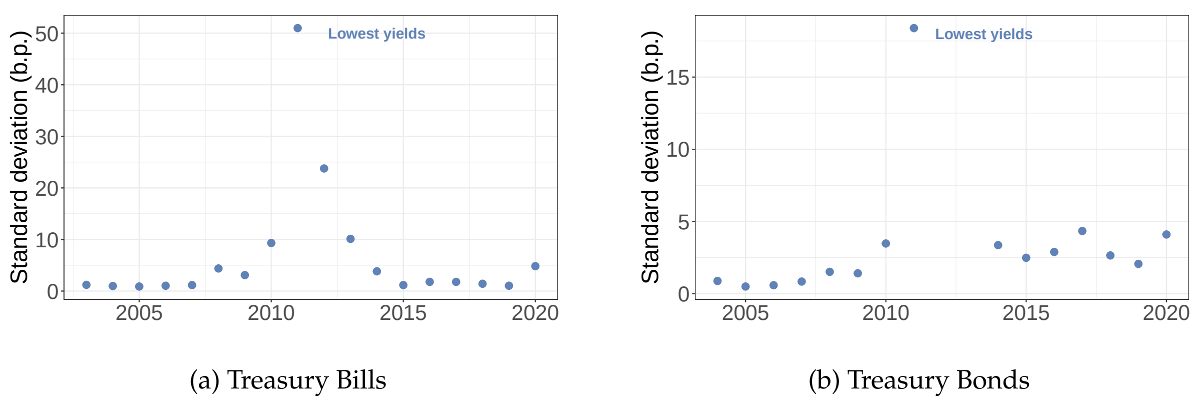


Figure 4: Variation between dealers' lowest bid yields

For both securities we see that prior to the crisis the standard deviation is very small. We then see a temporary increase during the crisis period followed by a return to zero afterwards. This pattern is more clear for treasury bills given the continued issuance of these securities during the crisis. Apart from that, the main difference with respect to treasury bonds is that after the crisis the variation does not quite go back to zero, instead it remains at slightly higher levels than before the crisis. This change in pattern occurs at the same time as the protocol for treasury bond auctions switched from discriminatory to uniform price.

Figure 5 presents the standard deviation of bid yields, within a bid function, for each dealer. More precisely, each point represents the average of such standard deviations across auctions, $\{1, \ldots, M_t\}$, for a given dealer, i, and a given year, t, as follows:

$$\overline{SD}_{i,t} = \frac{1}{M_t} \sum_{j=1}^{M_t} \sqrt{\sum_{k=1}^{K_j} \frac{(p_{ikj} - \overline{p}_{ij})^2}{N}}$$

The time series of the average standard deviation as a similar trend across all dealers (across panels): 1) increasing towards 2008; 2) a drop in 2009 before the crisis; 3) higher from 2010 to 2012; 4) a decrease starting in 2013 particularly accentuated in 2014; and, 5) almost flat bid functions from 2014 onward. Note, however, that some dealers have more disperse bid functions than others during the crisis period.

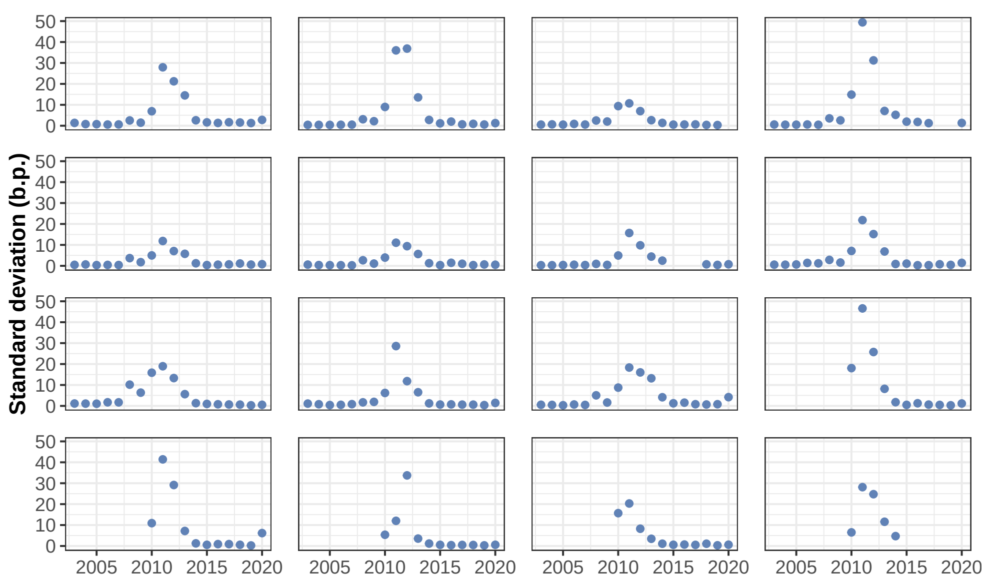


Figure 5: Average standard deviation of bids between dealers

Having said this, the standard deviation across and within investors is at roughly the same magnitude. This leads us to conclude that the pattern we see in Figure 3 is due to all investors bidding at a wider range of prices, as well as some of them bidding at higher prices and others at lower prices. That is, steeper aggregate bid functions are due, in part, to steeper individual bid functions.

No persistent heterogeneity. Finally, we evaluate whether there are persistent differences across investors. To do so, we rank the first bids, with the lowest yields, across dealers in a given auction. As before, we use the first bid as it is the most informative about differences across dealers. We postulate that if there were persistent differences across investors, we would see a persistent pattern in this ranking. For instance, well informed dealers would likely bid closer to the marginal price of the auction and consistently be ranked lower. Figure 6 depicts the relative ranking over time for each dealer across treasury bill auctions. We focus on treasury bills to highlight this fact due to the continued issuance of this type of securities during the crisis[12]. One can see that a persistent pattern does not seem to exist, in fact, ranking over time seems to be independent of dealer.

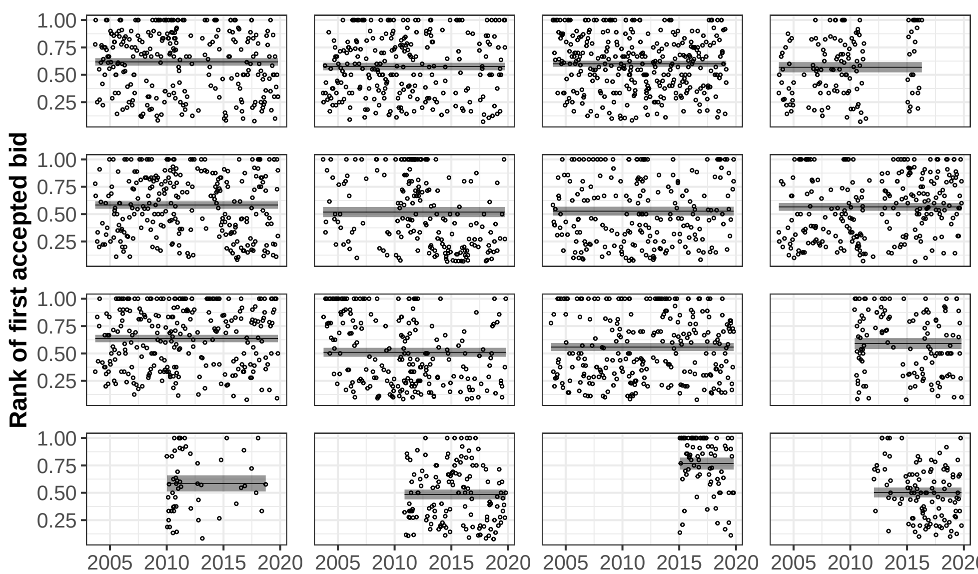


Figure 6: Rank of first bid (if accepted) over time

The horizontal bars in Figure 6 represent the dealer fixed effect, α_i, in the following regression:

$$R_{it} = \alpha_i + \epsilon_{it}$$

[12]The same pattern emerges for treasury bonds.

where the dependent variable R_{it} depicts dealers' i ranking in the auction ran at time t. Year fixed effects were not included due to lack of significance. The dealer fixed effects, as seen on the figure are fairly close to each other, with few exceptions, mostly on the lower panels. Particularly, those exceptions tend to be more significant for dealers that participate in auctions during shorter periods of time[13]. Overall, individual and time fixed effects account for less than 5% of the variation of rankings across investors and over time[14].

3.3 Government Spending Uncertainty

Unanticipated large government deficits were an important driver of the sovereign debt crisis in Europe. In late 2009, the newly elected Greek government disclosed that its budget deficits were far higher than previously though, this led to the downgrade of Greek debt and to a sharp increase in spreads. Copelovitch et al. (2016) consider this to be the event that triggered the Eurozone debt crisis. For Portugal in particular, the period leading to the bailout was marked by higher public spending and lower resources to finance that spending. These led to elevated borrowing that, together with a prolonged recession, played a role in explaining the sovereign debt crisis that followed.

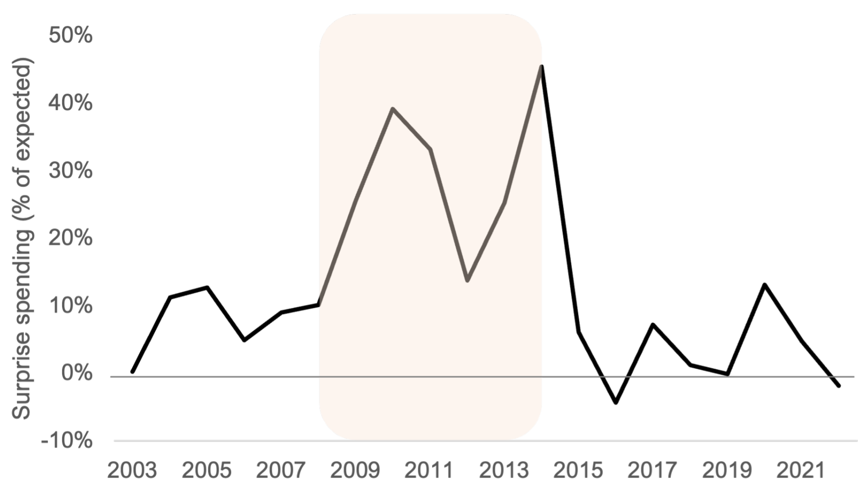


Figure 7: Deviation from expected public spending

[13]From all participating investors, 6 were not included in the plot as they participated for even shorter periods of time, making it harder to highlight trends in their ranking.

[14]We further test for linear trends within investors across time and verify that they are either not significant or explain less than 10% of the variation of the ranking over time.

Figure 7 highlights the increase in unanticipated public spending that became more evident after 2008. The figure plots the difference between expected spending and actual spending as a percentage of the expected spending. In particular, expected spending is taken from the government's spending proposal submitted at the end of the year for the year ahead. The figure shows that the deviations between expected and actual spending are mostly positive and go up to 45% above the 1 year ahead expectation. Noticeably, from 2008 and through 2014, the shaded area, these deviations were not only positive, but also higher than before and after the crisis.

3.4 Taking Stock

The evidence provided is going to discipline the model introduced in the next sections. In particular, we model uncertainty regarding financing needs, through public spending surprises, as these played an important role in Portugal and Southern Europe leading up to the crisis. The fact that information regarding these financing needs is asymmetric – it is privately observed by the government – will be the driving force leading to differences in bidding across protocols. This is consistent with the fact that supply of debt is random ex-ante – given the realization of the financing needs and after observing the prices demanded by investors, the government chooses how much to borrow optimally – the government reserves discretion on the quantities sold.

Given the evidence presented in section 3.2, as investors do not present persistent differences nor there are consistent differences in the levels of individual bid functions over time, as a simplifying assumption, we will consider that investors are symmetric[15].

[15] In fact, as it will become clear in the model, individual quantities bid by dealers need not be symmetric. Symmetry is used as a way to pin down the individual quantities bid. The fact that all investors bid the same price schedule, is a simplifying assumption that can be relaxed, with the difference being the introduction of aggregate demand uncertainty. For clarity of exposition and identification of the mechanisms affecting outcomes, we keep the symmetry assumption in what follows.

4 A Simple Model with Endogenous Borrowing

Before going into the quantitative model, we first start with a simple environment to illustrate mechanisms of a single auction environment. There are two periods, $t = \{0, 1\}$. The small open economy is populated by a government that borrows from a unit continuum of identical, competitive, risk neutral and deep pocketed foreign investors. These investors' discount factor is given by R^{-1}.

The government is benevolent and maximizes the welfare of the small open economy, endowed with y in each period. Preferences over streams of consumption are as follows:

$$\mathbb{E}\left[u(c_0) + \beta u(c_1)\right]$$

where u is strictly increasing and concave and $\beta \in (0, 1)$ denotes the discount factor.

In the first period, the economy faces a spending shock, θ, privately observed by the government. The spending shock is drawn from a discrete distribution with support on a finite set of points in the interval $[\theta_L, \theta_H]$ with $\theta_L < \theta_H$ and cdf G.

The government lacks commitment. In the second period an outside option, v^d, is realized. The outside option is drawn from a continuous distribution with support on $[\underline{v}, \overline{v}]$ and cdf F. We assume that $f(v^d) = F'(v^d) > 0$ on $[\underline{v}, \overline{v}]$. We further assume that $u(y) \geq \overline{v}$, so the government never defaults absent a strictly positive level of debt due.

Timing. The government starts the first period with endowment, y, and debt position, b_0. The spending shock, θ, is realized. Investors submit bid schedules. The government chooses optimally how much to borrow, given the bid schedules. In the second period the outside option, v^d, is realized and the government decides whether or not to default on its debt.

4.1 Auction Protocols

We consider the two types of protocols typically used for auctioning sovereign debt: the uniform price protocol (UP) and the discriminatory price protocol (DP). The protocol determines which bids are accepted by the government and at which prices they are executed.

Investors submit bid functions, a tuple $(p, b, K) = \left(\{p_k, b_k\}_{k\in\{1,...,K\}}\right)$ with $K < \infty$. A bid is a pair (p_k, b_k), representing the highest price p_k an investor is willing to pay to purchase b_k units of debt. The government sorts bids from highest to lowest (price) and accepts all bids until it is able to borrow b' units. The lowest accepted price, the one that clears the auction, is referred to as the marginal price of the auction, P_c.

Under UP, all accepted bids are executed at the same price, which corresponds to the marginal price of the auction, P_c. We analyse a the most common DP, the *"pay-as-bid"*, under which accepted bids are executed at the respective bid price. The auction protocol is chosen before the auction, and all bidders know it.

Denote the marginal price for a given ℓ as $P_c(\ell)$, where ℓ is the quantity issued. Then, the revenue under a uniform price protocol is simply:

$$\Delta(\ell)^U = P_c(\ell) \times \ell$$

On the other hand, given an aggregate bid schedule $p(b)$, revenue under a discriminatory price protocol is:

$$\Delta(\ell)^D = \int_0^\ell p(b)db$$

Refer to Figure 1 for a graphical representation of the two protocols and the respective revenues.

4.2 Optimal Bidding

In this environment, the private exogenous shock induces uncertainty on the amount of new debt the government will issue. As a result, investors submit multiple bids – a downward sloping individual demand.

We proceed to characterize optimal bidding. Let us first point out that it is never optimal to bid at a price that is not in the set of marginal prices. That is, it is never optimal to bid a price that is never chosen by the government.

Proposition 1. *Bidding marginal prices is a dominant strategy for investors.*

To see this, first recall that a bid is executed if the price is greater or equal than the realized marginal price $P_c(\theta; p)$. Consider two consecutive marginal prices $P_{c,1} > P_{c,2}$, and a bid with price p such that $P_{c,1} > p > P_{c,2}$. Note that such a bid is accepted with the same probability of a bid with $p = P_{c,2}$, that is $Pr[P_c(\theta) \leq p] = Pr[P_c(\theta) \leq P_{c,2}]$, as there is no marginal price between p and $P_{c,2}$. Moreover, the value of debt is independent of the price bid, as it is evaluated at the amount issued under the realized marginal price. The cost associated with bidding $p > P_{c,2}$, however, is greater: under a discriminatory price protocol, the cost is $p > P_{c,2}$. It follows that bidding marginal prices is a strictly dominant strategy under a discriminatory price protocol. Finally, we can consider the uniform price protocol as the limiting case of the discriminatory price protocol, where the cost of a winning bid is equal to the marginal price of the auction. In this case, bidding marginal prices is a weakly dominant strategy. Refer to the appendix for a full proof.

Given the result stated in proposition 1, as well as the fact that the state space (for θ) is discrete and so is the set of marginal prices, it follows that investors submit a finite number of bids, K, one for each possible realization of the marginal price. The lender problem is then to choose the quantities to bid for each price in the set of marginal prices. This result simplifies the nature of the problem. Instead of choosing quantities and prices, lenders are now only choosing quantities for each price in the set of prices that the government chooses with positive probability.

Lenders are infinitesimal. As such, an individual lender does not influence the aggregate amount issued in the auction nor the market clearing price of the auction. This is equivalent to say that each lender takes as given the aggregate bid function, as well as the government's strategy. Lenders' strategies are aggregated into a market demand curve. For each realization of θ, let $\ell(\theta; p)$ denote the total quantity issued in an auction, a point in the aggregate demand curve, evaluated at the marginal price $P_c(\theta; p)$. As in this setting debt has one period maturity, total debt and new issuances are the same object, $B' = \ell$. When a total of ℓ is sold, each unit has value $Q(\ell)$, the discounted future payment of a unit of debt. The payoff of submitting a bid function (p, b, K) is as follows:

$$\max_{b \in \mathbb{R}_+^K} \left\{ \sum_{k=1}^{K} \mathbb{E}_\theta \left[\underbrace{\mathbf{1}\{p_k \geq P_c(\theta)\}}_{\text{Prob. of winning bid}} \left(\underbrace{Q(\ell(\theta))}_{\text{Value per unit}} - \underbrace{\phi\big(p_k, P_c(\theta)|\ell(\theta)\big)}_{\text{Cost per unit}} \right) b_k \right] \right\}$$

where $\phi(p, P_c(\theta)|\ell(\theta))$ is the price paid by the lender to purchase a unit of the bond when they bid p and the marginal price is $P_c(\theta)$. In the two protocols, $\phi(\cdot)$ is such that $P_c(\theta) \leq \phi(p, P_c(\theta)|\ell(\theta)) \leq p$ and is weakly decreasing in p.[16] As investors are infinitesimal and b_k does not affect $\ell(\cdot)$ nor $P_c(\cdot)$, it follows that the payoff is separable over bids. Individual bids (p_k, b_k) solve:

$$\max_{b_k \geq 0} \mathbb{E}_\theta \left[\mathbb{1}\{p_k \geq P_c(\theta)\} \Big(Q(\ell(\theta)) - \phi(p_k, P_c(\theta)|\ell(\theta)) \Big) b_k \right]$$

In equilibrium, the expected payoff to a lender of any bids with $b_k^\star > 0$, must be 0. If this value were to be negative, then the lender could improve its payoff by setting b_k to 0. If that value were to be positive, then the lender could arbitrarily increase its payoff by setting b_k arbitrarily large. As such,

$$\mathbb{E}_\theta \left[\mathbb{1}\{p_k \geq P_c(\theta)\} \Big(Q(\ell(\theta)) - \phi(p_k, P_c(\theta)|\ell(\theta)) \Big) \right] = 0$$

This condition pins down the set of marginal prices. In particular, we must have:

$$0 = \int_{P_c^{-1}(p_k)}^{\theta_H} \Big(Q(\ell(\theta)) - \phi(p_k, P_c|\ell(\theta)) \Big) dG(\theta)$$

where $P_c^{-1}(p_k) \equiv \theta(p_k)$ denote the minimum θ that induces a marginal price of p_k.

In the case of a uniform price auction, this becomes:

$$0 = \int_{\theta(p_k)}^{\theta_H} \Big(Q(\ell(\theta)) - P_c(\ell(\theta)) \Big) dG(\theta)$$

[16]The functional form of $\phi(\cdot)$ represents the protocol being used in the auction, with this assumption we are allowing for a more general set of functional forms that nests both the uniform and discriminatory price protocols.

Since this must be true for every θ (including the largest one), this implies that for every θ which is in $supp\{G\}$:

$$P_c(\ell(\theta)) = Q(\ell(\theta))$$

and it follows that investors bid all possible realizations of the value of debt $Q(\ell(\theta))$. Prices are pinned down solely by the probability of default, as $Q(\cdot)$, the value of debt, equals its discounted expected future payments, which in this setting is simply the discounted probability of repayment.

For a discriminatory price auction, $\phi(p_k, P_c|\ell(\theta)) = p_k$ and

$$0 = \int_{\theta(p_k)}^{\theta_H} \Big(Q(\ell(\theta)) - p_k \Big) dG(\theta)$$

which can be simplified to:

$$p_k = \frac{1}{1 - G(\theta(p_k))} \int_{\theta(p_k)}^{\theta_H} Q(\ell(\theta)) dG(\theta) = \mathbb{E}[Q(\ell(\theta))|\theta \geq \theta(p_k)]$$

As such, prices are no longer uniquely pinned down by the probability of default. Investors commit to pay p_k regardless of how much the government borrows within the auction. In order to break even in expectation, the bid price, p_k, must be equal to the expected value of debt conditional on the bid being accepted. That is, investors bid according to their expectation of how much the government will borrow – prices depend on investors' beliefs about the government's borrowing distribution. Under a discriminatory price protocol, although investors break even in expectation, they may incur losses or profits ex-post.

Without loss of generality, because lenders are infinitesimal, we restrict consideration to symmetric pure strategy equilibria. Equilibria where every investor submits the same identical bids. This restriction pins down individual quantities b_k. With this, we abstract from the coordination problem between investors and focus on the strategic interaction between investors and the government.

4.3 Government

In the second period, the government chooses whether or not to default to solve:

$$W(B', v^d) = \max_{d \in \{0,1\}} \quad \left\{(1-d)u(y-B') + dv^d\right\}$$

The default policy rule is then:

$$d = \begin{cases} 1, \text{ if } \; v^d > u(y-B') \\ 0, \text{ if } \; v^d \leq u(y-B') \end{cases}$$

Define $\underline{v}^d(B')$ as the value of the outside option such that the government is indifferent between defaulting and repaying, for each B'. That is, $\underline{v}^d(B') \equiv u(y-B')$. Note that, $\underline{v}^d(B')$ is decreasing in B'. Moreover, the value of debt in period 0 is given by $Q(B) = R^{-1}F(\underline{v}^d(B))$, the discounted probability of repayment. Importantly, the function $Q(B)$ does not depend on the protocol used. That is, in this single auction environment, differences in bids across protocols do not arise due to different valuations of debt.

In the first period the government chooses borrowing, B', and clearing price, P_c, to solve:

$$U(B;p) = \max_{\{B' \geq 0, \; P_c \geq 0\}} \quad \left\{u(y + \Delta(B') - b_0 - \theta) + \beta\mathbb{E}\left[W(B', v^d)\right]\right\} \tag{1}$$

$$\text{s.t.} \quad B' = \ell(P_c)$$

$$\begin{cases} \Delta(B';p) = p(B')B', \text{ under UP} \\ \Delta(B';p) = \int_0^{B'} p(B)dB, \text{ under DP} \end{cases}$$

where $\ell(P_c) = \sum_{k=1}^{K} \mathbb{1}\{p_k \geq P_c\}b_k$.

Where the government takes the price schedule as given when maximizing utility. The optimal borrowing level, $B'(\theta;p)$, is a function of θ, given the price schedule. It follows that differences in prices may induce different borrowing decisions of the government.

4.4 Equilibrium

We are now ready to define an equilibrium in this environment. All the objects, and associated problems are defined above.

Definition 1 (Equilibrium). *Given the auction protocol, an equilibrium consists of values* $\{U, W\}$, *a price equation,* Q, *a bid function,* p, *and policy rules,* $\{d, B, P_c\}$, *such that:*

1. *The price equation equals the discounted probability of repayment, given policy rules;*
2. *The bid function satisfies ex-ante zero profits for investors, given policy rules and prices;*
3. *The policy rules solve the government's problems given values and prices;*
4. *The auction clears, given the bid function and policy rules.*

4.5 Static Trade-offs and Revenue Equivalence

For a clear exposition of the trade-offs between the protocols, let us first consider an example with exogenous borrowing. That is, B' is not chosen by the government, but rather its realization is exogenously random. In particular let us look at the case where the distribution is known by all, but the realization is privately observed by the government. Let both B' and v^d follow a uniform distribution between 0 and y. Figure 8 and 9 below, illustrate the bid schedules and revenue, respectively, under this setting.

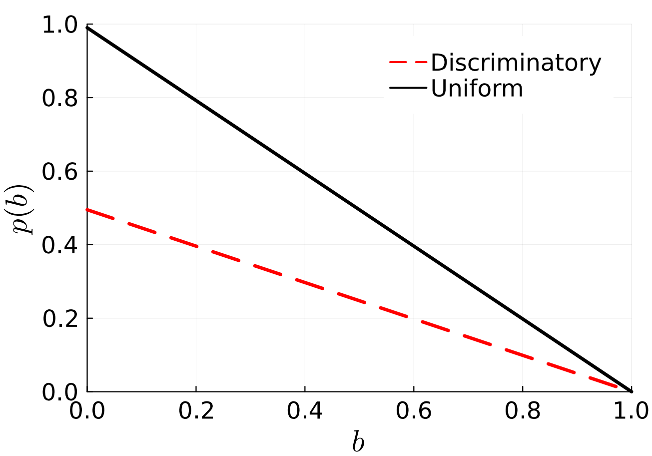


Figure 8: Aggregate bid functions under UP and DP

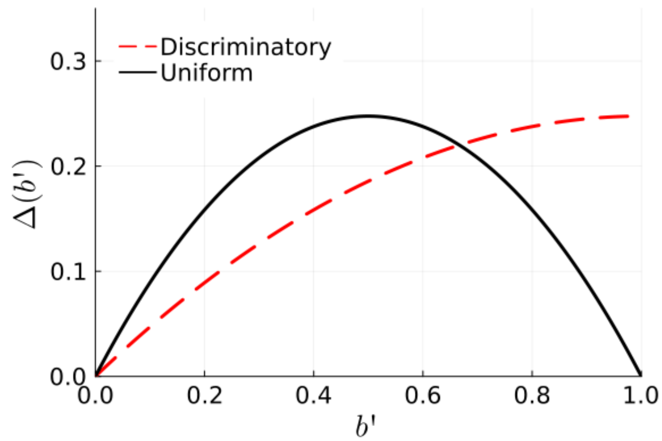


Figure 9: Revenue under Uniform and Discriminatory auctions

There are two main takeaways. Firstly, under a discriminatory price protocol, the prices at each increment are smaller (or equal, at the last increment) than the prices under a uniform price protocol. Secondly, revenue is higher (lower) under a discriminatory price protocol (uniform price protocol) for high (low) levels of B'. This follows from the fact that in a uniform price protocol the marginal price is also the average price, as opposed to the discriminatory price protocol, where the average price is above the marginal price.

Let us first focus on the difference in levels between the two schedules. Note that, in a discriminatory price protocol, investors commit to paying what they bid, even if the marginal price of the auction is lower than that. This commitment to pay a price above the marginal price, introduces what we refer to as static dilution. The costs associated with static dilution under a discriminatory price protocol are responsible for lower prices at each increment being borrowed.

Consider the following analogy. There is one borrower and several lenders that live in individual houses. The government knocks randomly at a lender's door and asks her how much she requires to lend an increment of debt b_1. The lender offers a price such that she breaks even in expectation. After borrowing the first increment, with a positive probability, the government will knock at someone else's door. If it does so, and borrows more, the value of debt goes down as $Q(B)$ is decreasing in B. The first investor sees her position in the asset diluted by the increase in borrowing that the government partakes in. The first investor knows at the time of lending, however, that this dilution would happen with a given positive probability. As such, she requires a higher return (lower price) when asked to lend in the first place. In particular, she will bid a price that equals her expectation of the value of the asset, conditional on the amount the government is already borrowing. This is true at each increment that the government decides to borrow, which explains the lower prices under a discriminatory price auction, as long as there is positive probability that the government will keep on borrowing. Since this probability is decreasing as we move along the bid schedule, dilution in the marginal price is also decreasing. That is, at the margin, first bids (higher prices, lower amounts) are more impacted by this dilution.

In the analogy above we described what seemed to be a game of sequential borrowing, reminiscent of Bizer and DeMarzo (1992). Note, however, that the government interacts simultaneously with all investors participating in an auction. Investors submit bid functions accounting for all possible realizations of B', and dilution takes place across states of the world (realizations of B') and not across time. Hence the name, static dilution.

Let us now focus on the differences in revenue across states. Under a uniform price protocol, revenue is given by the product of the marginal price of the auction and the quantity borrowed. Under a discriminatory price protocol, revenue is given by the sum of prices at each infinitesimal increment, the area below the curve.

Figure 9 shows that, although revenue is lower under a discriminatory price protocol for low levels of B', it is weakly increasing in B'. On the other hand, revenue under a uniform price protocol yields the familiar Laffer curve where, for high levels of B', the decrease in price more than offsets the increase in quantity and, ultimately, revenue goes back to zero. In a discriminatory price protocol, although the marginal price is decreasing in B', all previous increments are executed at their bid prices. In a uniform price protocol, however, all increments are issued at the lowest price. As such, the discriminatory price protocol provides insurance by allowing the government to transfer resources from good states (low B') to bad states (high B').

The choice of protocol highlights a mean-variance trade-off: the discriminatory price protocol has lower marginal prices, but provides insurance through lower variance of average prices. This is a similar point to Cole et al. (2018), where the choice of protocol depends on how the government values different states of the world.

In this example, as B' is exogenous, the utility in the second period is independent of the auction protocol. It follows that welfare is pinned down by the utility in the first period and, in particular, by the auction revenue. Let $\hat{\Delta}(B')$ and $\Delta(B')$ denote revenue under a uniform and a discriminatory price protocol, respectively.

Theorem 1 (Revenue Equivalence)**.** *If B' is a random variable independent of the auction protocol, then ex-ante expected revenue in the auction is the same under both protocols.*

This result was expected and stems from the investors break-even condition: i) as B' is independent of the protocol, so will be the investor's expected payoff; ii) as investors break even in expectation and i) holds, the expected cost of winning bids must also be independent of the protocol; iii) note that investors' expected cost is the government's expected revenue. Refer to the appendix for a formal proof.

Revenue equivalence tells us that a risk neutral government is indifferent between protocols as the ex-ante expected revenue is independent of the protocol. Note however that, a risk averse government prefers the discriminatory price protocol, as consumption is less volatile and has the same expected value. The lower volatility of consumption under DP follows from revenue having a smaller variance, both because of the lower variance in prices and the fact that revenue itself is weakly increasing. As, $y - b_0$ is a constant and consumption is given by $y - b_0 + \Delta(b)$, consumption is less volatile under DP. Revenue equivalence gives us that expected consumption is the same across protocols.

Note however that, with endogenous borrowing, the main difference is that optimal borrowing $B'(\theta, p)$ depends on the price schedule, and so it depends on the protocol chosen. This fact, induces, through θ, different distributions for the issuance decision. As $B'(\theta; p)$ is not independent from the protocol chosen, revenue equivalence need not hold. To see this, note that $\mathbb{E}[B'Q(B')]$ may be different across protocols as the distribution of the decision rule, $B'(\theta; p)$, may differ across protocols. Then, in order for investors to break even in expectation, the cost of winning bids (and revenue for the government) is no longer equal across protocols.

This result highlights the role of the government's strategic behavior. Introducing a government that chooses debt optimally, and does so after prices are submitted, is enough to break revenue equivalence.

4.6 Numerical Example with Endogenous Borrowing

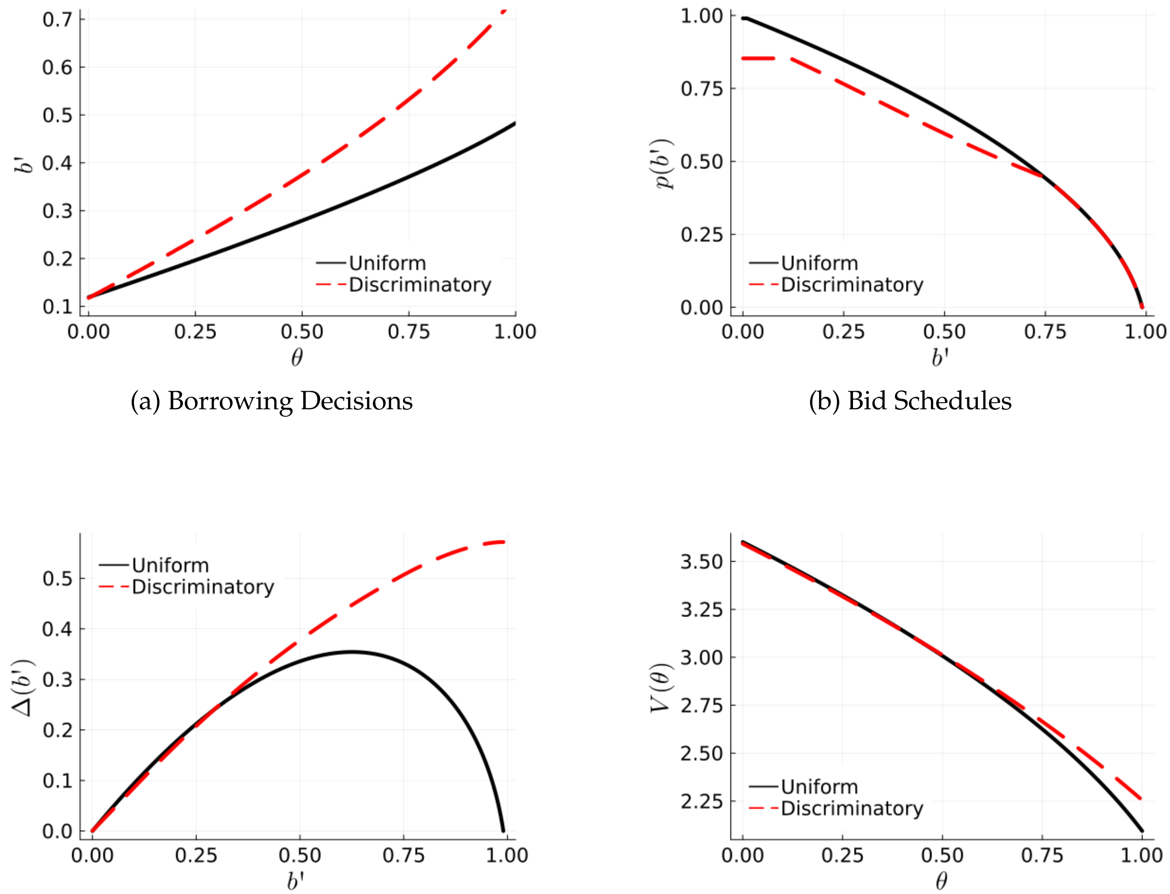
We now present a numerical exercise highlighting the differences in borrowing decisions induced by the differences in prices detailed above. At the same time, we highlight the overall impact of the protocol on revenue and welfare.

We let θ be exponentially distributed with cdf $G(\theta) = 1 - \exp(-\lambda\theta)$[17]. We set $\lambda = 4$. Further, let v^d be uniformly distributed on the interval $[0, y]$. Then, $F(v^d) = v^d/y$ and $F'(v^d) = (1/y)$. We further set $y = 1$, $\beta = 0.9$, $R = 1.01$ and $b_0 = 0$. Finally, we assume the functional form of utility to be constant relative risk aversion (CRRA). In particular, we will consider $\gamma = 0.5$ and *log* utility ($\gamma \to 1$).

Figure 10 below depicts how outcomes compare under CRRA utility with $\gamma = 0.5$. In panel (a) we see that under DP the government borrows more than under UP, and that this difference is increasing in θ. This relates to the bid schedules depicted in panel (b) and the revenue curves depicted in panel (c): 1) at the first bids, prices are much lower under DP due to a higher static dilution at the margin; 2) static dilution at the margin decreases along the bid schedule; and 3) the price moves against the government under UP as all debt is issued at the marginal price, decreasing the incentive to borrow at the margin when compared to DP. Ex-ante welfare is higher under UP:

$$\mathbb{E}[V(\theta)_{UP}] = 3.329 \quad \mathbb{E}[V(\theta)_{DP}] = 3.327$$

[17] In practice, we use a discretized and truncated exponential distribution that preserves the assumption in the theory.



(a) Borrowing Decisions

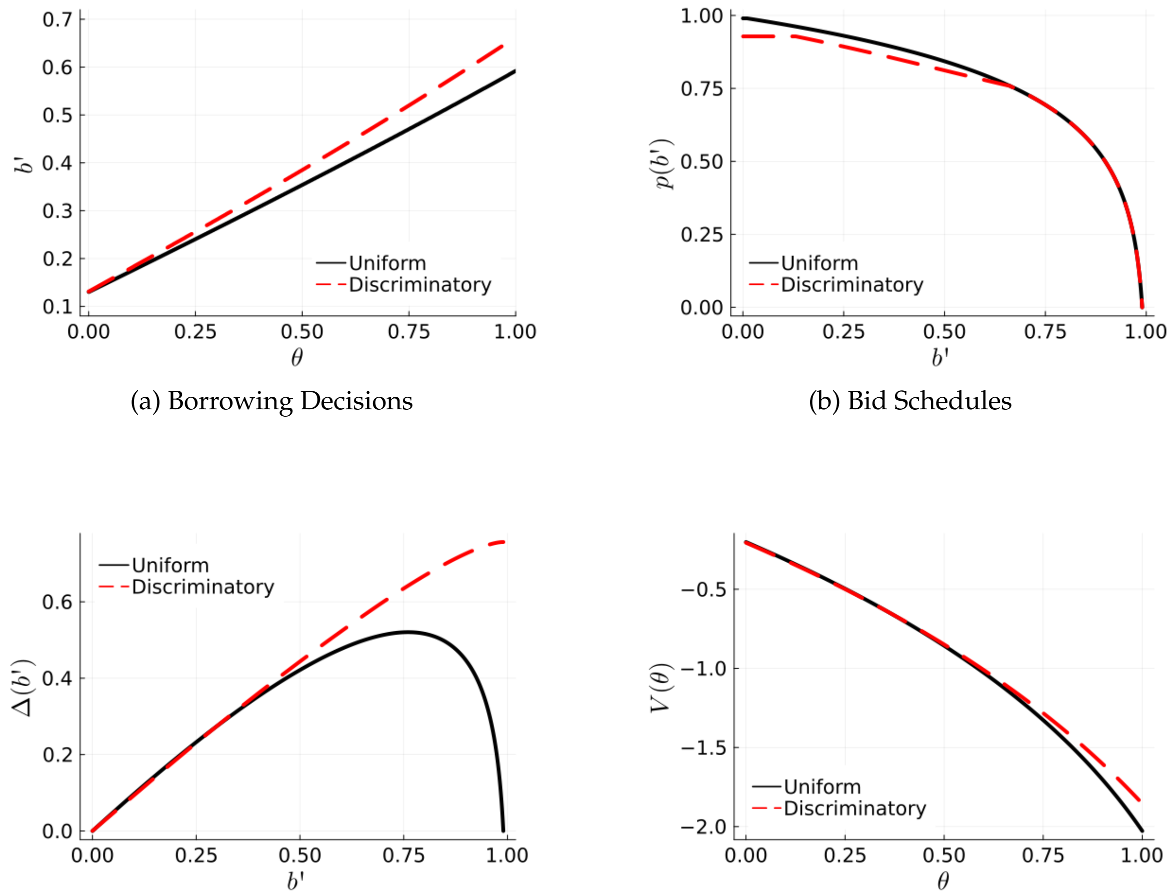
(b) Bid Schedules

(c) Revenue

(d) Value functions

Figure 10: Comparing optimal outcomes under UP and DP

A point that is worth highlighting. Although it was expected that under DP the government would borrow more than under UP, as $\gamma \to 0$ the government borrows as much as possible and does so faster than it would for higher values of γ. This illustrates that static dilution increases as the government preferences move towards risk neutrality. At the same time, insurance benefits are valued less. Both forces lead to the uniform price protocol being preferred by a government with a low coefficient of relative risk aversion.



(a) Borrowing Decisions

(b) Bid Schedules

(c) Revenue

(d) Value functions

Figure 11: Comparing optimal outcomes under UP and DP

In Figure 11, we see how the outcomes compare when we instead use *log* utility. In panel (a) we see a behavior for borrowing similar to the one described in Figure 10 . The important difference is that with the added concavity the government borrows less than before. The added concavity reinforces the importance of consumption smoothing across states and across time. It introduces a disciplining effect on borrowing that limits static dilution. Panel (b) depicts the effect of this added discipline on the price schedule. Less dilution implies higher prices under DP than those observed with $\gamma = 0.5$. Panel (d) shows that the discriminatory price protocol now fares better than under the previous example.

In fact, ex-ante welfare is now higher under the discriminatory protocol:

$$\mathbb{E}[V(\theta)_{UP}] = -0.500 < \mathbb{E}[V(\theta)_{DP}] = -0.498$$

To assess whether this pattern is not dependent on a specific set of functional forms and parameterizations, we look at different degrees of risk aversion, distributions of θ and v^d and different patterns of output growth across periods. For a list of different functional forms and parameterizations refer to the appendix.

4.7 Taking Stock

So far we have used a simple single auction environment to understand how the choice of protocol affects borrowing decisions and the cost of debt. To isolate differences on the bid schedules we used an example with exogenous borrowing. We can summarize the results as follows: 1) bid prices are lower under DP than under UP due to static dilution; 2) bid prices have less variance across B' under DP; 3) auction protocols are revenue equivalent as B' is exogenously random and investors break-even in expectation; 4) a risk neutral government is indifferent between protocols; and 5) a risk averse government prefers the discriminatory price protocol as expected revenue is the same and variance is smaller.

With endogenous borrowing we get that: 1) auction protocols are not revenue equivalent; 2) for each θ, the government borrows more under the discriminatory price protocol; 3) the difference in borrowing is increasing in θ as revenue keeps increasing and static dilution at the margin decreases; 4) under a sufficiently concave utility, the need to smooth consumption across time disciplines borrowing and limits the costs associated with dilution under the discriminatory price protocol. As a result, the latter fares better than the uniform price protocol.

5 A Quantitative Model with Long Bonds and Repeated Auctions

We now introduce dynamics by considering an environment with repeated auctions and long-term debt. This environment allows us to evaluate the interaction of the auction protocols with debt dilution across time and how it affects borrowing, the cost of debt and welfare. As described before, the choice of protocol hinges on a trade-off. One can think of this as a trade-off between commitment and insurance. The discriminatory price protocol provides insurance across states, but the fact that revenue is non-decreasing induces the government to over-borrow relative to the uniform price protocol. That is, the uniform price protocol provides more commitment to the government which, in anticipation, leads to higher average prices. The interaction of repeated auctions with long bonds is crucial to assess how this trade-off operates: the government will have an incentive to over-borrow in every auction going forward under a discriminatory protocol and this is priced by investors when submitting bids.

We start by calibrating the model to the Portuguese economy under the discriminatory price protocol, the one being used prior to the crisis. We then perform a counterfactual, solving the model as if the uniform price protocol was the one being used. With both specifications, the one under the discriminatory price protocol, and the counterfactual, under the uniform price protocol, we then assess how protocols interact with default risk and how they affect auction outcomes and welfare.

5.1 Environment

Time is discrete and infinite, $t = \{0, 1, 2, \dots\}$. The small open economy is populated by a government that borrows from a unit continuum of competitive, risk neutral and deep pocketed foreign investors. These investors' discount factor is given by R^{-1}.

The government is benevolent and maximizes the welfare of the small open economy.

Preferences over streams of consumption are as follows:

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

where u is strictly increasing and strictly concave and $\beta \in (0,1)$ denotes the discount factor.

There is a public exogenous state of the world $s \in \mathcal{S}$, which follows a Markov process that governs the endowment $y(s)$ and expected public spending $g(s)$. The private exogenous state of the world includes $T \in \mathcal{T}$, the government's type, that is i.i.d over time, and determines a budget surprise θ_T.

The government borrows using a defaultable long term bond. We assume that there is a finite set $\mathcal{B}$ of values that the government's debt level B can take. We follow Chatterjee and Eyigungor (2012) and Hatchondo and Martinez (2009), and model debt as a contract promising a stream of exponentially declining coupon payments. Specifically, at time t, a unit of the bond promises to pay $(1-\lambda)^{t+j-1}(\lambda+\kappa)$ of the consumption good in period $t+j$. As in the two period model, in order to issue debt B' the government runs an auction and investors provide bid schedules.

If the government chooses to default, the country is excluded from financial markets and suffers a flow utility cost of $h(s)$. The country regains access to financial markets with probability η. Reentry is done through restructuring. That is, upon reentry the government is liable for a fraction $(1-\tau)$ of the debt, B, it had prior to the default event.

If a country enters a period in good standing, the timing is as follows:

1. The states s, $\mathbf{m} = \left\{\{m^R(B')\}_{B\in\mathcal{B}},\ m^D\right\}$ and T are realized.
2. The government chooses whether to default or not.
   - 2.1. If the government has chosen to repay $(d=0)$:
     - i. The government runs an auction;
     - ii. Investors submit bid functions after observing the public state s;

iii. The government chooses B' and P_c, given the aggregate bid function.

2.2. If the government chooses to default ($d = 1$), it is excluded from financial markets and cannot borrow.

i. Next period, with probability η the government regains access to financial markets, and with probability $(1 - \eta)$ remains excluded;

Let us note that the private exogenous state of the world does not need to be a shock that occurs every time the government runs an auction. That is, the frequency of the shock does not need to be linked to the auction calendar. The auction is, however, the only opportunity investors have to know more about the realization of the private shock. There is no other way the shock gets observed unless an auction takes place. This follows from the fact that the government does not participate in the secondary market. Even if we thought of a shock to the market's ability to absorb more debt, the same argument would hold. Investors cannot infer that from the secondary market, as debt is in fixed supply until a new auction takes place. When an auction takes place investors then know if there was, in fact, a new realization of the shock.

5.1.1 Additional Elements

The private exogenous state includes a vector **m** of preference shocks for the government that is i.i.d. over time. These preference shocks enter additively in the government's decision problems. They are unbounded and therefore ensure that every feasible action is played with positive probability in equilibrium. Introducing these shocks is like introducing randomization, ensuring convergence – that an equilibrium exists [18]. These shocks are otherwise small.

There is an issuance cost $i(s, B, B')$ incurred when the government adjusts its debt level. This is a standard feature in models with long term debt and positive recovery rates (see Dvorkin et al. (2021) or Chatterjee and Eyigungor (2015)). Without these adjustment costs, the government has an incentive to issue very large amounts of debt when default is

[18] These preference shocks have the same role as the m shock introduced in Chatterjee and Eyigungor (2012).

imminent in order to extract the value of existing bondholders' securities. This type of "maximum" dilution behavior is counterfactual. As such, issuance costs are added to the model to prevent it from occurring in equilibrium. Quantitatively, the amount spent financing the issuance costs ends up being small.

5.2 Optimal Bidding

The solution to the lenders' problem is similar to what we described in the two period environment: each lender takes as given the strategies of all the other lenders, as well as the government's strategy. Lenders' strategies are aggregated into a market demand curve. For each realization of $(T, \mathbf{m})$, let $\ell(T, \mathbf{m}; p)$ denote the total amount issued in an auction, a point in the aggregate demand curve, evaluated at the marginal price $P_c(T, \mathbf{m}; p)$. Note that $\ell(\cdot)$ and $P_c(\cdot)$ are a function of the private state, for a given public state comprised of s and the current period debt level B. For this subsection, we suppress the current public state for ease of notation. When a total of ℓ is sold, the government's outstanding debt position is $B' \equiv \ell + (1 - \lambda)B$ and the value is $Q(\ell + (1 - \lambda)B)$, where $Q(\cdot)$ will be determined in equilibrium.

The result stated in Proposition 1 follows, with lenders bidding only marginal prices. Lenders' bid functions consist of K price quantity pairs, where the finiteness of K relies on the the finiteness of $\mathcal{B}$, the set of values that the government's debt level can take. As investors are infinitesimal and b_k does not affect $B'(\cdot)$ nor $P_c(\cdot)$, it follows, as before, that the payoff is separable over bids. Individual bids (p_k, b_k) now solve:

$$\max_{b_k \geq 0} \mathbb{E}\left[\underbrace{\mathbf{1}\{p_k \geq P_c(T, \mathbf{m})\}}_{\text{Prob. of winning bid}} \left(\underbrace{Q(B'(T, \mathbf{m}))}_{\text{Value per unit}} - \underbrace{\phi\big(p_k, P_c(T, \mathbf{m}) | B'(T, \mathbf{m})\big)}_{\text{Cost per unit}} \right) b_k \right]$$

In the two protocols, $\phi(\cdot)$ is such that $P_c(T, \mathbf{m}) \leq \phi(p_k, P_c(T, \mathbf{m}) | B'(T, \mathbf{m})) \leq p_k$ and is weakly decreasing in p_k. An equilibrium with positive b_k we must have:

$$0 = \mathbb{E}\left[\mathbb{1}\{p_k \geq P_c(T, \mathbf{m})\} \Big(Q(B'(T, \mathbf{m})) - \phi(p_k, P_c(T, \mathbf{m}) | B'(T, \mathbf{m})) \Big) \right]$$

This zero profit condition pins down the set of marginal prices. In the case of a uniform price auction, this becomes:

$$0 = \mathbb{E}\left[\mathbb{1}\{p_k \geq P_c(T, \mathbf{m})\}\Big(Q(B'(T, \mathbf{m})) - P_c(T, \mathbf{m})\Big)\right]$$

Since this must be true for every pair $\{T, \mathbf{m}\}$ (including the one that yields the smallest marginal price), this implies that for every pair $\{T, \mathbf{m}\}$:

$$P_c(T, \mathbf{m}) = Q(B'(T, \mathbf{m})) \tag{2}$$

For a discriminatory price auction, we instead obtain:

$$0 = \mathbb{E}\left[\mathbb{1}\{p_k \geq P_c(T, \mathbf{m})\}\Big(Q(B'(T, \mathbf{m})) - p_k\Big)\right]$$

which can be simplified to:

$$\begin{aligned} P_c(T, \mathbf{m}) &= \frac{\mathbb{E}\Big[\mathbb{1}\{P_c(T, \mathbf{m}) \geq P_c(\widehat{T}, \widehat{\mathbf{m}})\}\; Q(B'(T, \mathbf{m}))\Big]}{Pr[P_c(T, \mathbf{m}) \geq P_c(\widehat{T}, \widehat{\mathbf{m}})]} \\ &= \mathbb{E}[Q(B'(T, \mathbf{m}))|P_c(T, \mathbf{m}) \geq P_c(\widehat{T}, \widehat{\mathbf{m}})] \end{aligned} \tag{3}$$

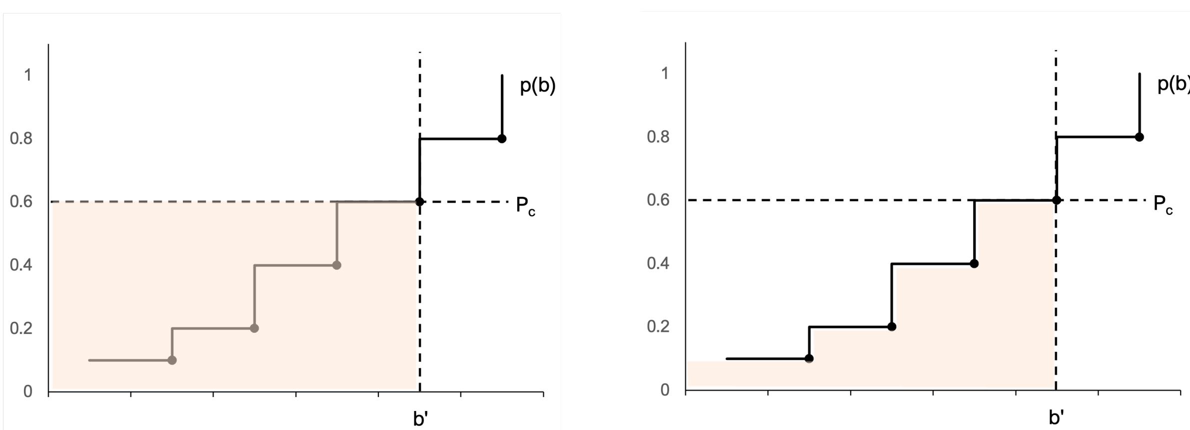
Note that, as before, under a uniform price auction prices are pinned down by the value of debt at each state, whether under a discriminatory price protocol prices depend on investors' beliefs about the government's borrowing distribution.

Once again, without loss of generality, because lenders are infinitesimal, we restrict consideration to symmetric pure strategy equilibria, equilibria where every investor submits the same identical bids. As before, this restriction pins down individual quantities b_k, and we abstract from the coordination problem between investors.

5.2.1 A Note on Reverse Auctions

In this environment, we allow the government to perform debt buybacks. This is done through reverse auctions, that are conducted under the same protocol as the auctions. Our underlying assumption is that the government either borrows or buys back debt in a given auction. That is, the government cannot borrow and buy back debt in the same auction. This is consistent with what we observe in the data, the government either runs an auction or a reverse auction at a give moment in time. Figure 12 below, depicts how the protocols work for reverse auctions.

Bids by primary dealers are represented by the increasing step function – the aggregate bid function. In a reverse auction the aggregate bid function is akin to a supply function. The government selects the amount it buys back, ℓ, and the clearing price, P_c. In a uniform price reverse auction, all accepted bids are executed at the marginal price. In a discriminatory price auction (pay-as-bid), all accepted bids are executed at the respective bidding prices. The cost to the government (negative revenue) is depicted by the shaded areas.



(a) A marginal price reverse auction

(b) A "pay-as-bid" reverse auction

Figure 12: Comparison of marginal price and pay-as-bid reverse auctions

Note that which auction protocol yields a higher cost to the government is not obvious since, as in the regular auctions, the aggregate bid function is different depending on the protocol used. In particular, under a discriminatory price reverse auction investors bid according to:

$$P_c(T,\mathbf{m}) = \frac{\mathbb{E}\Big[\mathbb{1}\{P_c(T,\mathbf{m}) \leq P_c(\widehat{T},\widehat{\mathbf{m}})\}\ Q(B'(T,\mathbf{m}))\Big]}{Pr[P_c(T,\mathbf{m}) \leq P_c(\widehat{T},\widehat{\mathbf{m}})]} \tag{4}$$

$$= \mathbb{E}[Q(B'(T,\mathbf{m}))|P_c(T,\mathbf{m}) \leq P_c(\widehat{T},\widehat{\mathbf{m}})]$$

which is the counterpart of equation (3) for regular auctions. The uniform price protocol for reverse auctions, yields the same optimal bidding as in equation (2), with marginal prices equal to the value of debt.

From equations (3) and (4) we see that, under a discriminatory price auction, optimal bidding has: i) investors bidding weakly below the value of the asset when the government is issuing debt; ii) investors bidding weakly above the value of the asset when the government is buying back debt.

5.3 Government's Problem

At the beginning of the period, if the country is in good standing, the government's problem is to choose whether or not to default as follows:

$$V\,(s,\mathbf{m},T,B) = \max_{d\in\{0,1\}} \Big\{(1-d)V^R(s,\mathbf{m^R},T,B) + d\left(V^D(s,T,B) + m^D\right)\Big\}$$

where V^R is the repayment value function and V^D is the value under default.

The value under default is given by:

$$V^D(s,T) = (1-\beta)\ \big[u\big(y(s) - \underbrace{g(s)\times\theta_T}_{\substack{\text{Realized}\\\text{Spending}}}\big) - h(s)\big] +$$
$$+\,\beta\left(\mathbb{E}\Big[\eta V\,(s',\mathbf{m}',T',(1-\tau)B)) + (1-\eta)V^D(s',T',B)\Big|s\Big]\right)$$

where the government does not have access to financial markets and just consumes what is left of the endowment after financing realized public spending, the product of the expected $g(s)$ and the surprise θ_T. The default cost, $h(s)$, is measured in utils. The continuation value depends on whether the government regains access to financial markets, which occurs with probability η. In such cases, the government is liable for a fraction $(1-\tau)$ of the debt it had due prior to the default event. Otherwise, with probability $(1-\eta)$ the government remains in default.

Conditional on choosing to repay its debt, the government's problem is:

$$V^R\left(s, \mathbf{m^R}, T, B\right) = \max_{\{c\geq 0,\ P_c>0,\ B'\in\mathcal{B}\}} \left\{(1-\beta)\, u\left(c\right) + \beta\mathbb{E}\Big[V\left(s', \mathbf{m}', T', B'\right)\Big| s\Big] + m^R(B')\right\}$$

$$\text{s.t.}\quad c + (\lambda+\kappa)B + \underbrace{g(s)\times\theta_T}_{\substack{\text{Realized}\\ \text{Spending}}} = y(s) + \underbrace{\Delta\Big(s, B, \ell(s,B,P_c)\Big)}_{\text{Auction Revenue}}\left(1 - i(s,B,B')\right)$$

$$B' = \underbrace{\ell(s,B,P_c)}_{\text{Debt Issuance}} + (1-\lambda)B$$

where $\Delta(\cdot)$ denotes the auction revenue (or reverse auction cost) and $\ell(\cdot)$ the new debt issuances (or debt buyback), as follows:

$$\Delta(s,B,\ell(s,B,P_c)) = \begin{cases} \sum_{k=1}^{K} \mathbb{1}\{p_k(s,B)\geq P_c\}\ \phi(p_k(s,B),P_c)\ b_k(s,B)\ , & \text{if } B' \geq (1-\lambda)B \\ \sum_{k=1}^{K} -\mathbb{1}\{p_k(s,B)\leq P_c\}\ \phi(p_k(s,B),P_c)\ b_k(s,B)\ , & \text{if } B' < (1-\lambda)B \end{cases}$$

$$\ell(s,B,P_c) = \begin{cases} \sum_{k=1}^{K} \mathbb{1}\{p_k(s,B)\geq P_c\} b_k(s,B)\ , & \text{if } B' \geq (1-\lambda)B \\ \sum_{k=1}^{K} -\mathbb{1}\{p_k(s,B)\geq P_c\} b_k(s,B)\ , & \text{if } B' < (1-\lambda)B \end{cases}$$

In this problem, the optimal choice of debt holdings for next period, B', depends on debt issuances, ℓ. These, for a given bid schedule, are pinned down by the marginal price of the auction, P_c. As such, the government chooses both P_c and B'. The resulting auction revenue together with the endowment are used to finance pubic consumption and to service debt (coupon and principal repayments), with the remaining going towards consumption. When choosing borrowing and consumption, the government takes into account how its choices affect both the revenue raised today and the continuation value it will receive in the future $\mathbb{E}[V(\cdot)|s]$.

Before we proceed, it is worth discussing how the value of debt changes across protocols. In the two period environment, we had $Q(b) = R^{-1}F(u(y-b))$, the discounted probability of repayment, for both protocols. Here however, conditional on repayment, the government chooses a new B' every period and so the value of debt is not necessarily the same across protocols. In fact, given a protocol j, the value of debt is given by its discounted expected payments, as follows:

$$Q_j(s,B') = R^{-1}\,\mathbb{E}\left[(1-d_j')\Big((\kappa+\lambda)+(1-\lambda)Q_j(s',\mathcal{B}_j(s',B',\mathbf{m}',T'))\Big) + d_j' Q_j^D(s',B')\Big|s\right]$$

where, $d_j' = d_j(s',m',T',B')$ denotes states where the government defaults and $(1-d_j')$ the states where the government repays its debt ; $(\kappa+\lambda)$ denotes the coupon payment plus the repayment of the fraction of debt λ that matures; $(1-\lambda)Q(s',\mathcal{B}(s',B',\mathbf{m}',T'))$ the continuation value of the fraction of debt that does not mature; and, $Q^D(s',B')$ the residual value of debt upon default.

$$Q_j^D(s,B) = R^{-1}\left(\eta(1-\tau)Q_j(s,(1-\tau)B) + (1-\eta)\mathbb{E}\Big[Q_j^D(s',B)|s\Big]\right)$$

where, with probability η the government regains access to financial markets and restructuring materializes in a haircut τ on the debt stock the government had prior to the default episode; with the complement probability $(1-\eta)$ the government remains excluded from financial markets.

From these equations, it follows that $Q(s,B')$ is potentially different across protocols, both because: 1) the continuation value of investors' debt claims are potentially different as different bid schedules likely induce different borrowing decisions, $\mathcal{B}(\cdot)$; and 2) default decisions are likely different as well: for the same shock realizations, $V^R(\cdot)$ is potentially different both because of differences in borrowing but also through differences in revenue across protocols (both level and variance). These differences in value, Q, induced by differences in borrowing and default decisions highlight the effects of repeated auctions. This dynamic channel translates into different valuations for the same asset (a claim on government's future payments), depending on the protocol.

At the same time, this general environment also introduces a dynamic inefficiency, dynamic dilution, through the use of long maturity debt. When issuing additional debt in the future, original investors see the value of their claims fall as the probability of default increases. The dynamic nature of the general environment gives us this new channel. The crucial aspect here is how the protocol interacts with debt dilution. In particular, what are the incentives on borrowing that each protocol provides and what are the corresponding effects on prices, default and welfare.

5.4 Equilibrium

We are now ready to define an equilibrium in this environment. All the objects, and associated problems and functional equations, are defined above.

Definition 2 (Equilibrium)**.** *Given the auction protocol, a recursive equilibrium consists of value functions,* $\{V,\ V^R,\ V^D\}$*, price equations,* $\{Q, Q^D\}$*, bid function* p*, and policy rules,* $\{d, B, \ell, P_c\}$*, that satisfy the following conditions (for the full, detailed list, see the appendix):*

1. *The price equations satisfy their functional equations, given policy rules;*
2. *The bid function satisfies ex-ante zero profits for investors, given policy rules and prices;*
3. *The policy rules solve the government's problems, given values and prices;*
4. *The value functions satisfy their functional equations given prices and policies;*
5. *The auction clears, given the bid function and policy rules.*

6 Calibration

In this section, we specify the data used for the estimation. We further describe the functional forms used in the quantitative implementation of the model and detail the calibrated parameter values. After that, we show how well the model fits the data.

6.1 Data

We use data from the Portuguese economy to perform a case study for the theory developed in this paper. As described in section 3, we have detailed data on Portuguese sovereign debt auctions. Furthermore, by using Portugal as an example we can evaluate the switch in protocol that occurred in the midst of the sovereign debt crisis. Finally, Portugal fits the core assumptions of this type of sovereign default model as it is a small open economy with a vast majority of its debt securities held by foreigners.

Annual data for real and nominal GDP is taken from Eurostat national accounts and covers the period 1995-2022. Monthly data on long-term interest rates for Portugal and Germany is taken from the European Central Bank (ECB) Interest Rate Statistics, covering the same period, 1999-2022. Government debt securities data is taken from BPStat general government statistics. Finally, we use annual data for realized government expenditures and revenues as well as the government's one year ahead expectation for those same rubrics for the period 2003:2022. Realized expenditures and revenues are obtained from the *Portuguese Public Finance Council* (CFP, Portuguese acronym). The year ahead estimates are obtained from the government's budget proposal reports, submitted every year in October.

In the model, a period is a year. This choice is primarily due to the annual frequency of the data for the expected public spending.

In what follows, we select the structural parameters of the model. Some of these are set independently outside of the model, from the literature. We estimate the income and expected spending process. We measure spending deviations in the data and parametrize a log normal distribution for the spending surprise process. The remaining parameters are calibrated by simulated method of moments to match certain characteristics of the Portuguese economy.

6.2 Functional Forms and Parameters

The model is calibrated to match the experience of Portugal since joining the Euro. The annual risk-free real interest rate, r is set to 0.02, a standard value, close to the average rate in germany in the relevant period. The maturity rate λ of the bond and its coupon value κ are set to the values used by Paluszynski (2023) (who also studies Portugal during the same period).

The functional form of utility is constant relative risk aversion:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

We set the relative risk aversion coefficient, γ, to 2, a standard value in macroeconomics. The preference shocks $\mathbf{m}$ are distributed according to a Generalized Type One Extreme Value distribution with scale parameter σ_m and correlation parameter ρ_m. These distributions are chosen for their computational tractability[19].

We estimate the income process for detrended real per capita GDP for years 1995:2019 and the process for the detrended year ahead expectation of the real per capita public spending for years 2003:2019, both assumed to be AR(1) as follows:

$$y_t = \mu_y + \rho_y y_{t-1} + \epsilon_t$$
$$g_t = \mu_g + \rho_g g_{t-1} + \nu_t$$

In doing so, we compute the correlation between the estimated innovations, $corr(\epsilon_t, \nu_t) = \rho_{\epsilon,\nu}$. Finally, we get σ_θ, as the standard deviation of the log differences between the actual detrended real public spending and the detrended real year ahead expectation for the same variable. With that, we parameterize the spending surprise shock as follows:

$$\theta_t \sim \text{log-normal}\,(0, \sigma_\theta)$$

[19]Specifically, both choice probabilities and ex ante expected values can be written analytically in terms of the values associated with the choices. We set ρ_m following Dvorkin et al. (2021). We then set the scale parameter at a small number that still ensures convergence, half of that of Dvorkin et al. (2021).

Other functional forms that must be specified are the cost of default and the issuance cost function. The utility cost of default is as follows:

$$h(y_t) = \max\{0, (1-h_0) + h_1 \log y_t\}$$

The issuance cost function is as in Fourakis (2023)[20]. This function imposes a strict limit on the one period ahead default probability from which issuing costs are positive and is continuous in the scale of the issuance. The purpose of these issuance costs is to prevent a behavior Chatterjee and Eyigungor (2015) termed "maximum dilution." When default is imminent, the maturity structure of the debt, together with the opportunity for restructuring, gives the government an incentive to issue as much debt as possible, extracting the value of existing bondholders'. Issuance cost functions counteract these incentives.

The remaining parameters are calibrated by simulated method of moments. The idea behind the identification strategy is to match certain general characteristics of the Portuguese economy. The targeted moments are the mean of the external debt to GDP ratio while not in default and the mean and volatility of spreads while not in default.

The three parameters β, h_0 and h_1, govern the average impatience of the government and the average penalty for defaulting. These parameters are closely tied to the mean of the debt to GDP ratio and the mean and volatility of the interest rate spreads. In particular, β controls the rate at which the government accumulates debt up to levels where default may occur, and h_0 and h_1 determine how high such levels are. Moreover, the default cost parameters control how likely the government is to default for every state. As such, they are directly related to the spreads, which are a measure of the risk of default. Tables 2 and 3 summarize the calibrated parameters and those set outside the model.

[20] A sine wave shifted and scaled to rise from 0 to 1 as it travels from the threshold, p_d, to 1:

$$i\left(s,B,B'\right) = \begin{cases} 0 & B' \leq \hat{B} \text{ or } \Pr\left(d'^{\star}=1\right) \leq p_d \\ \frac{1}{2}\left(1+\sin\left(\pi\left(\frac{\Pr(d'^{\star}=1)-p_d}{1-p_d}-\frac{1}{2}\right)\right)\right) & B' > \hat{B} \text{ and } \Pr\left(d'^{\star}=1\right) > p_d \end{cases}$$

where $\hat{B} = \max\{(1-\lambda)B, 0\}$.

Table 2: Calibrated parameters

Parameters	Value
β	0.933
h_0	0.917
h_1	0.325

Table 3: Parameters set independently

Parameters	Value	Source
R	1.02	Standard
γ	2	Standard
λ	0.212	Paluszynski (2023)
κ	0.050	Paluszynski (2023)
η	0.154	Chatterjee and Eyigungor (2012)
μ_y	0.005	Estimation
ρ_y	0.802	Estimation
σ_ϵ	0.019	Estimation
μ_g	-0.388	Estimation
ρ_g	0.773	Estimation
σ_ν	0.054	Estimation
$\rho_{\epsilon,\nu}$	0.397	Estimation
σ_θ	0.115	Estimation
ρ_m	0.25	Dvorkin et al. (2021)
σ_m	$5e-4$	For convergence
τ	0.535	Greek haircut
p_d	0.8	-

6.3 Targeted Moments

Table 4 below compares the targeted moments for calibration in the data, for 1999Q1 – 2011Q1, and in the model. The calibrated model is able to closely match the targeted moments in the data.

Table 4: Targeted moments

Moments	Data	Model
$\mathbb{E}[b'/y]$	48.91%	46.22%
$\mathbb{E}[r - r_f]$	0.61%	0.63%
$\sigma(r - r_f)$	1.02 p.p.	1.03 p.p.

Where r, in the model, is the internal rate of return that makes the present discounted value of the promised sequence of future payments on a unit bond equal to the unit price, computed as follows:

$$r(s, B') = \frac{(\lambda + \kappa)}{Q(s, B')} - \lambda$$

It is worth stressing that, as documented in Aguiar et al. (2016), standard sovereign debt models typically fail at matching the volatility of the spread. In particular, volatility tends to be lower in the model than in the data. Even though Portugal presents spreads with higher volatility than the average spread, the model is able to match it. It is worth noting that Portuguese sovereign debt auctions were using the discriminatory price protocol during the period for which the model was calibrated. The discriminatory price protocol is prone to incentivize higher marginal spreads. As the decrease in price (increase in spread) only affects the marginal unit borrowed, the government is willing to borrow further into higher spreads. This is in opposition to the uniform price auction where a decrease in price (increase in spread) affects the whole debt issuance. It turns out that this incentive to borrow further captures both higher spreads and most importantly, the spread volatility that standard models fail to attain. Paluszynski (2023) calibrates a standard sovereign borrowing and default model and finds that such model cannot generate the volatility of the spreads observed in the data. The author then introduces unobserved rare disasters into the output process. This addition to the model generates a spread volatility that surpasses that observed in the data[21]. The author justifies this increase in volatility as follows: A "sudden fall in the belief may cause a downward shift of the en-

[21]This is the case for conditional simulations. For long run simulations high volatility of spreads comes at the expense of counterfactual debt and spreads.

tire bond price schedule [...] spreads may then shoot up, while income remains relatively high, making default unattractive because of the nonlinear punishment function". That is, the government is willing to sustain higher spreads instead of defaulting right away as it would if the income process was a standard AR(1). Notice that both modeling decisions yield a similar result, of higher volatility of the spreads. Here we highlight that for this particular modeling decision we rely on an institutional feature observed in the data, while the remaining modeling decisions are fairly standard. Furthermore, both Aguiar et al. (2016) and Paluszynski (2023) were doing a calibration for countries that used the discriminatory price protocol (Mexico and Portugal, respectively). The use of the right auction framework is enough to offset the highlighted difficulty.

6.4 Validation

So far we have described the data that identifies the key parameters of the model as well as the calibration strategy. We now move to validation, in particular assessing the model performance with respect to matching of untargeted moments. To do so, we also perform a counterfactual exercise, by solving the model as if the protocol used was the uniform. As such, we not only see how well the discriminatory performs but also highlight differences to the uniform price protocol specification.

Table 5 below presents the simulated business cycle moments in the long-run sample (i.e. ergodic distribution), under both protocols, along with the empirical moments[22].

Here we present two new spreads, the average bid spread and the average spread on the last bid accepted. These use, respectively, r_{bid} and r_{marg}, that are computed similarly to r as follows:

$$r_{bid}(s, B, B') = \frac{(\lambda + \kappa)}{\overline{p}(s, B, B')} - \lambda$$

[22] Model moments are generated from simulations that extend to 10,000 years and are repeated 1,000 times. Empirical moments involving spreads are computed using annual data and average spreads from 1999 to 2010, the year before the bailout. Empirical moments using average bid spreads were computed for treasury bond auctions. Other empirical moments are computed using annual data starting from 1995 and up to 2019.

$$r_{marg}(s,B,B') = \frac{(\lambda+\kappa)}{p(s,B,B')} - \lambda$$

where $\overline{p}(s,B,B')$ denotes the average price of executed bids in an auction and $p(s,B,B')$ denotes the price of the last accepted bid in an auction.

Recall that the model is calibrated for the discriminatory price protocol, and the simulated moments under the uniform price protocol are counterfactual. As such, the average debt to GDP ratio, the average spread and the volatility of the spread are matched under the discriminatory price protocol but not under the uniform price protocol. Noticeably, the average spreads under the uniform protocol are much lower and less volatile than under the discriminatory protocol. Moreover, under the uniform protocol, the volatility of the spreads is lower than the average spread, contrary to the empirical moments. This observation goes back to the aforementioned discussion regarding the limitation of standard models in inducing volatile spreads as seen in the data. The average spread on the last bid accepted, $\mathbb{E}[r_{marg} - r^\star]$, highlights the willingness to borrow more on the margin under this protocol. The difference between the average spread in the secondary market, $\mathbb{E}[r - r^\star]$, and the average bid spread, $\mathbb{E}[r_{bid} - r^\star]$, highlights the static dilution in the discriminatory price auction.

Table 5: Moments of the Ergodic Distribution

	Data	Discriminatory	Uniform
$\mathbb{E}[r - r^\star]$	0.61%	0.63%	0.26%
$\mathbb{E}[r_{bid} - r^\star]$	0.79%	0.66%	0.26%
$\mathbb{E}[r_{marg} - r^\star]$	0.82%	1.01%	0.26%
$\sigma(r - r^\star)$	1.02 p.p.	1.03 p.p.	0.15 p.p.
Default Rate	-	0.97%	0.43%
$\mathbb{E}\left[b'/y\right]$	48.91%	46.22%	50.64%
$\sigma(tb/y)$	4.35 p.p.	2.70 p.p.	2.24 p.p.
$\sigma(c)/\sigma(y)$	1.49	1.54	1.54
$corr(tb/y, y)$	-0.48	-0.05	-0.10
$corr(tb/y, r - r^\star)$	0.18	-0.10	-0.06
$corr(y, r - r^\star)$	-0.54	-0.22	-0.34
$corr(y, r_{marg} - r^\star)$	-0.76	-0.32	-0.34
$corr(y, r_{bid} - r^\star)$	-0.76	-0.54	-0.34

The model, under the discriminatory price protocol, generates underpricing compared to the secondary market and over-payment within the auction, across bids, as in the data. On the one hand, investors require a higher spread (lower price), on average, in the auction than the one present in the secondary market, a phenomenon usually referred to as underpricing. On the other hand, on average, investors overpay in the first bids, as the average spread across bids is lower than the spread on the marginal bid (the average price investors pay is higher than the price paid for the marginal bid). These are more pronounced in the model than in the data.

Even though the average spread in the primary market is higher than the one in the secondary market, investors break even in expectation. That is, on average, the cost of buying debt is the same as the cost of reselling debt on the secondary market:

$$\mathbb{E}[\Delta(s, B, B')] = \mathbb{E}[Q(\cdot)(B' - (1-\lambda)B)]$$

where, $\Delta(s, B, B')$ is the revenue for the government and, as such, the cost to investors.

This equivalence is not recovered in the spreads for two reasons. First, the average of the spreads, the inverse of the price, $\mathbb{E}[1/p]$, is not the ratio of the expectations. Second, the weight of each accepted bid in determining the average price of debt sold varies with the amount of debt sold, and so *even the average prices differ between primary and secondary markets*. For clarity, the average price is recovered as revenue over quantity issued as follows[23]:

$$\begin{aligned}
\overline{p}(s, B, B') &= \frac{\Delta(s, B, B')}{B' - (1-\lambda)B} \\
&= \frac{\sum_{j=1}^{K} b_j(s, B)\ p_j(s, B)}{B' - (1-\lambda)B} \\
&= \frac{\sum_{j=1}^{K} b_j(s, B)\ \mathbb{E}[Q(s, B'(s, B, j))|B^* \geq B'(s, B, j)]}{B' - (1-\lambda)B}
\end{aligned}$$

[23]To simplify notation we suppress the private states and use the implicit distribution for the optimal borrowing decision.

$$= \sum_{j=1}^{K} \underbrace{\frac{b_j(s,B)}{B' - (1-\lambda)B}}_{\text{Weight of bid } j} \frac{\sum_{n=j}^{N} Pr[B^* \geq B'(s,B,n)]\ Q(s,B'(s,B,n))}{Pr[B^* \geq B'(s,B,j)]}$$

From the expression above, one can see that the weight of bid j depends on the amount issued. These weights impact average prices and as a result it need not be the case that average prices are equal in primary and secondary markets, respectively $\overline{p}(s,B,B')$ and $Q(s,B')$. Summing up, we recover the ordering of spreads observed in the data, between primary and secondary markets, while keeping the underlying assumption that investors break even in expectation.

The supercharged incentive to dilute existing bondholders under the discriminatory price protocol prevents the government from sustaining a level of debt as high as under a uniform price protocol. The higher spreads observed under the discriminatory price auction are consistent with the higher default frequency also present under the discriminatory auctions. We opted to not estimate a default rate in the data as we believe there is no consistent way of doing so. In recent times there are no default events in Portugal. One could argue that without a bailout Portugal might have defaulted during the sovereign debt crisis, but using this episode alone would give us a very noisy estimate of the frequency of default.

In the model, y, $G = g \times \theta$ and $tb = y - c - g \times \theta$ all have clear meanings, respectively, GDP, public consumption and the trade balance. As detailed before, we have disciplined G and y directly using data on actual government consumption and GDP, respectively. In the national accounts of our economy this leaves us only one degree of freedom, tb or c. Since the mapping to the trade balance is far clearer, we choose that one. It leaves c as a residual mapping to the sum of consumption and private investment. As such, moments regarding c in the table above, both in the model and in the data, refer to the implied residual. This explains why the relative volatility of the implied residual to GDP is above 1, as investment is more volatile than GDP. Both the model under the discriminatory price protocol and the counterfactual are able to generate the high relative volatility of the implied residual observed in the data. Regarding the volatility of the trade balance, both specifications yield a less volatile trade balance than the one observed in the data, with the discriminatory specification doing better than the uniform one.

The model presents almost no correlation between the trade balance and neither output nor spreads. The biggest miss is on the correlation between the trade balance and output, substantially negative in the data. Finally, the last two rows present the correlations of spreads and output. In particular, we look both at spreads in the secondary market and the average spread across bids in a given auction. The model underestimates the absolute value of the correlations. Note, however, that it captures the relative correlation. As in the data, output has a stronger correlation with the average spreads in the primary market than with the spreads in the secondary market. This feature could not be matched under a uniform price protocol, or the standard model.

Before proceeding, it is worth returning to the discussion regarding the volatility of the spreads. Aguiar et al. (2016) mention that standard models ability to increase the volatility of the spreads relied on sufficiently high variability of output. The drawback it seemed, was that it "comes at the expense of tying the spread much too closely to output fluctuations." Here however, that is not the case as the calibrated model generates the spread volatility observed in the data while inducing a correlation between spreads and output that is close, but smaller, than the one observed in the data.

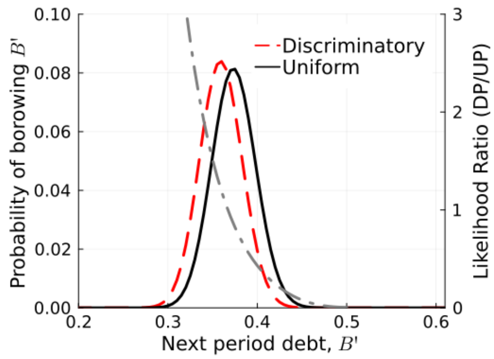
7 Comparing Protocols

In the following section we characterize some of the key properties of the equilibrium keeping the counterfactual exercise where we compare the auction protocols. We present the results of the model under a discriminatory price protocol, the one being used in Portugal during the period for which we calibrated the model. We then solve the calibrated model under a uniform price protocol. This is counterfactual as the uniform price protocol was not being used.

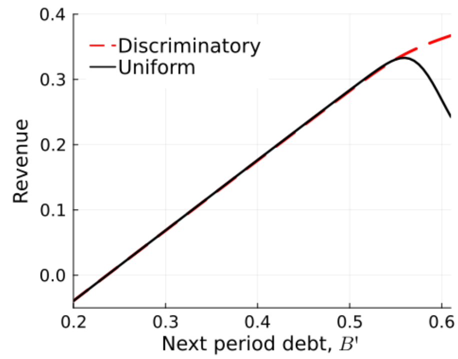
In this environment, due to the private exogenous state of the world, instead of a single valued policy function for borrowing decisions, investors infer a probability distribution over choices of borrowing. Figure 13 presents a comparison of these distributions, as well as revenue under the two protocols. Panels (a), (c) and (e) present the policy function distribution for next period debt holdings, B', for current debt $B \in \{0.3, 0.55, 0.65\}$ and at the average net output level $(y - g)$. Panels (b), (d) and (f) present the corresponding revenue at the same levels of current debt and net output.

For relatively low levels of current debt (panels (a) and (b)), the government borrows less under a discriminatory price protocol than under a uniform price protocol. Since this borrowing region has virtually no default risk, prices under both protocols are similar and relatively close to the risk free price. As the government keeps on borrowing, however, prices under the discriminatory protocol start decreasing sooner (for lower values of debt) and faster. It follows that, while the marginal benefit of an extra unit of debt (marginal revenue) is very similar across protocols, the marginal cost of an extra unit of debt, the change in the continuation value, is higher under the discriminatory price protocol. As such, the government borrows less than under a uniform price protocol. This argument hinges on the fact that for low initial values of debt, as is the case, it is not optimal for the government to borrow up to a point where the likelihood of default increases substantially. That is, for the government to be willing to borrow more under a discriminatory price protocol, it must be the case that marginal revenue is higher than that under a uniform protocol. This would only happen if the prices were decreasing considerably which, given the distribution of the private state, is not the case in the calibration.

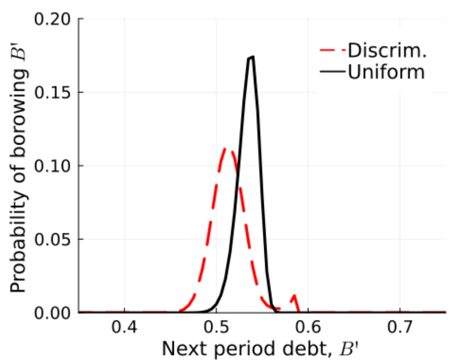
As we increase the initial debt level the conclusions are different. In particular, for $B = 0.55$ (panels (c) and (d)), we see that the discriminatory price protocol has the government choosing a wider range of borrowing levels with positive probability, when compared to the counterfactual uniform price protocol. This means that, for some realizations of the private state, the government borrows more than under a uniform protocol.



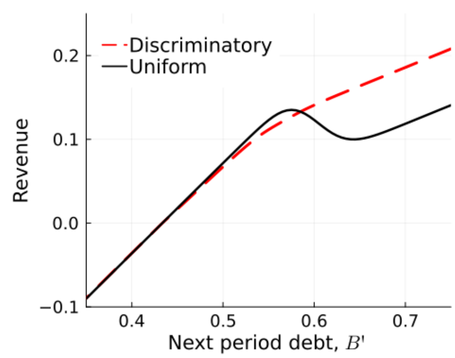
(a) Borrowing Decisions at $B = 0.3$ and $y = \bar{y}$



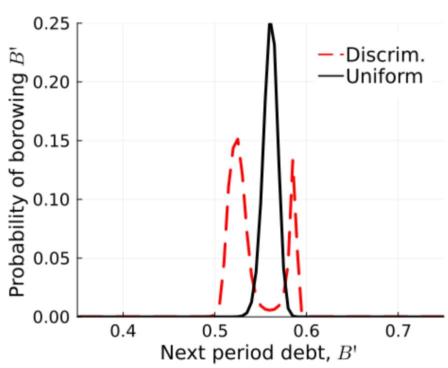
(b) Revenue at $B = 0.3$ and $y = \bar{y}$



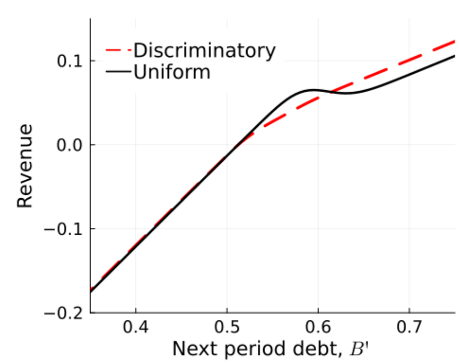
(c) Borrowing Decisions at $B = 0.55$ and $y = \bar{y}$



(d) Revenue at $B = 0.55$ and $y = \bar{y}$



(e) Borrowing Decisions at $B = 0.65$ and $y = \bar{y}$



(f) Revenue at $B = 0.65$ and $y = \bar{y}$

Figure 13: Comparing outcomes under UP and DP

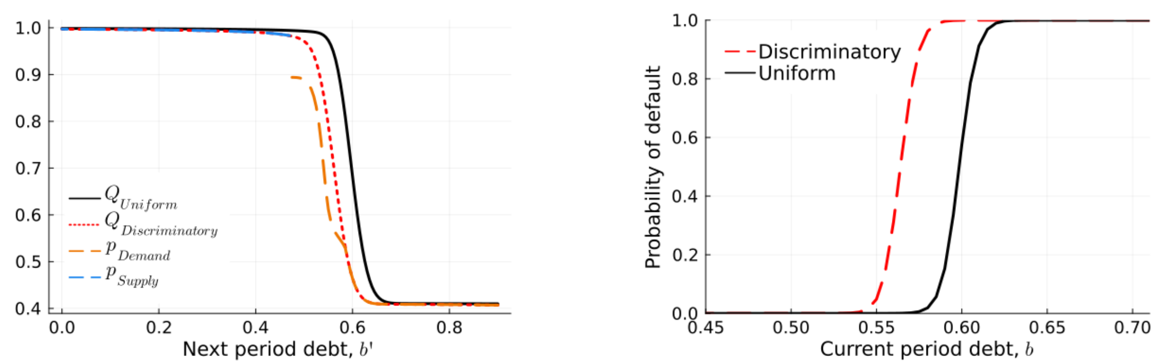
The higher initial debt means that with positive probability it is optimal for the government to borrow up to a region where the likelihood of default increases. As a result, investors submit lower prices under a discriminatory price protocol due to static dilution. This effect is highest, at the margin, for the first increments the government borrows, as investors expect the government to borrow more with positive probability. This can be seen in panel (d) where revenue under the discriminatory price protocol is lower than under the uniform when the government starts borrowing (after $B' = (1-\lambda)B = 0.55 \times (1-0.212) = 0.43$). Additionally, note that not only is revenue lower but so is marginal revenue. Under a uniform price protocol for these borrowing levels, marginal revenue is flat as prices are relatively stable and close to the risk free price. This comparison between prices and revenue rationalizes the fact that the government borrows lower amounts with a higher probability under a discriminatory price protocol.

As revenue is weakly increasing under the discriminatory price protocol as opposed to the counterfactual, the government has an incentive to borrow more under a discriminatory protocol, for some realizations of the private state. This is particularly relevant when the government starts the period highly indebted and is faced with negative surprise spending. In such cases: 1) revenue under the uniform protocol is decreasing; and, 2) static dilution under the discriminatory protocol, is decreasing at the margin. These two forces together rationalize the difference in borrowing at the right tail. The marginal benefit of borrowing remains positive under the discriminatory protocol whereas it becomes negative under the uniform. Under the discriminatory price protocol the incentive to borrow more and dilute existing bondholders is supercharged.

Finally, in panels (e) and (f), for initial levels of debt very close to inducing default with probability equal to one, the differences between protocols are exacerbated. The forces at play are the same, but as captured by the revenue curves, the differences are more significant. As the government gets closer to default, there is a higher likelihood of borrowing into a region with very low prices. It follows that both static dilution and the differences between marginal revenue across protocols are exacerbated. As a result, one of two things happen: 1) a good realization of the private state leads the government to borrow less as static dilution is at its maximum in this region; and 2) a bad realization of the private state leads the government to use the insurance properties of the discriminatory price protocol and borrow much more than under the uniform protocol as depicted in panel (e).

It is important to note that under the discriminatory price protocol, the government is not able to sustain as much debt as in the counterfactual under a uniform price protocol. As such, even when the discriminatory price protocol is accumulating less debt than the counterfactual, for low initial values of debt, it is using a higher fraction of the debt that it can sustain. This point becomes more apparent as we look at default decisions.

We next turn to the value of debt and bid schedules and how they relate to the default decisions, all depicted in Figure 14 below.



(a) Bid schedule at $y = \bar{y}$

(b) Default decision at $y = \bar{y}$

Figure 14: Comparing prices and default decisions under UP and DP

Panel (a) depicts the value of debt and bid schedules under a discriminatory price protocol, as well as under the counterfactual uniform price protocol. First, we see that the value of debt, $Q(\cdot)$, is different across protocols: it is weakly lower under a discriminatory price protocol. As mentioned, the difference between the value of debt across protocols highlights the impact of the incentives to borrow over time provided by the different protocols, the dynamic channel. Essentially, the discriminatory price protocol supercharges dynamic dilution. The government has an incentive to borrow more than under a uniform price protocol, as revenue is always increasing. Investors internalize the fact that this incentive is going to be present in every future auction, that is, they expect higher debt accumulations with positive probability under the discriminatory price protocol. As a result, the value of debt claims is lower.

Panel (a) also depicts the bid schedules under the two protocols. Recall that the bid schedule under a uniform price protocol overlaps with the value of debt. Under a discriminatory price protocol, we recover the same relationship between value and bids found in the two period environment. That is, within an auction, the discriminatory price protocol introduces static dilution with investors bidding weakly below the value of debt as long as they believe the government will keep on borrowing (within the auction) with positive probability. p_{supply} denotes the bid schedule for debt buybacks, with investors bidding weakly above the value of debt as long as they believe the government will keep on buying back (within the reverse auction) with positive probability.

Ultimately, these figures highlight that, when default risk is a concern, the uniform price protocol protects against static dilution within an auction and, at the same time, provides better incentives on borrowing, leading to lower dynamic dilution. The lower prices under a discriminatory price protocol depicted in panel (a) are consistent with the probabilities of default depicted in panel (b). Under a discriminatory price protocol, the probability of default is weakly higher than under a uniform price protocol – the default set is larger under the discriminatory price protocol.

7.1 Government's Welfare

We first discuss how we compare welfare across the protocols. Note that the relative contribution of the preference shocks to the value functions depends on the number of choices available to the government. As such, we consider values net of preference shocks throughout this section.

Given an initial state s_0 and debt B_0, we define the government's welfare as the expected discounted utility over surprise spending states and preference shocks , as well as future public states of endowment net of expected public spending. For example, the value under a discriminatory price protocol, for initial public state (s_0, B_0) is:

$$\mathbb{E}_{T,\mathbf{m},s_1}\left[V^{DP}(s_0, T, \mathbf{m}, B_0)\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t (1-\beta)\big(u(c_t^{DP}) - d_t^{DP} h(s)\big) \,\Big|\, s_0\right]$$

where $c_t^{DP} \equiv c^{DP}(s, T, \mathbf{m}, B)$ denotes equilibrium consumption and $d_t^{DP} \equiv d^{DP}(s, T, \mathbf{m}, B)$ denotes the government's equilibrium default decision.

We then compute the percentage increase in the consumption path, under a discriminatory price protocol, that would make the government indifferent between this allocation and the allocation where the government follows its optimal borrowing plan under a uniform price protocol. That is, we compute the equivalent variation in permanent consumption (ζ). Since utility is CRRA with relative risk aversion coefficient $\gamma \neq 1$ we can define ζ as in the following equation:

$$(1+\zeta)^{1-\gamma}\mathbb{E}\left[V^{DP}(s_0, \mathbf{m}, T_0, B_0)\right] = \mathbb{E}\left[V^{UP}(s_0, \mathbf{m}, T_0, B_0)\right]$$

Solving for ζ yields:

$$\zeta = \left(\frac{\overline{V}_{UP}}{\overline{V}_{DP}}\right)^{\frac{1}{1-\gamma}} - 1$$

where $\overline{V}_{UP}$ and $\overline{V}_{DP}$ are, respectively, the average value under a uniform price protocol and a discriminatory price protocol for an initial state (s_0, B_0).

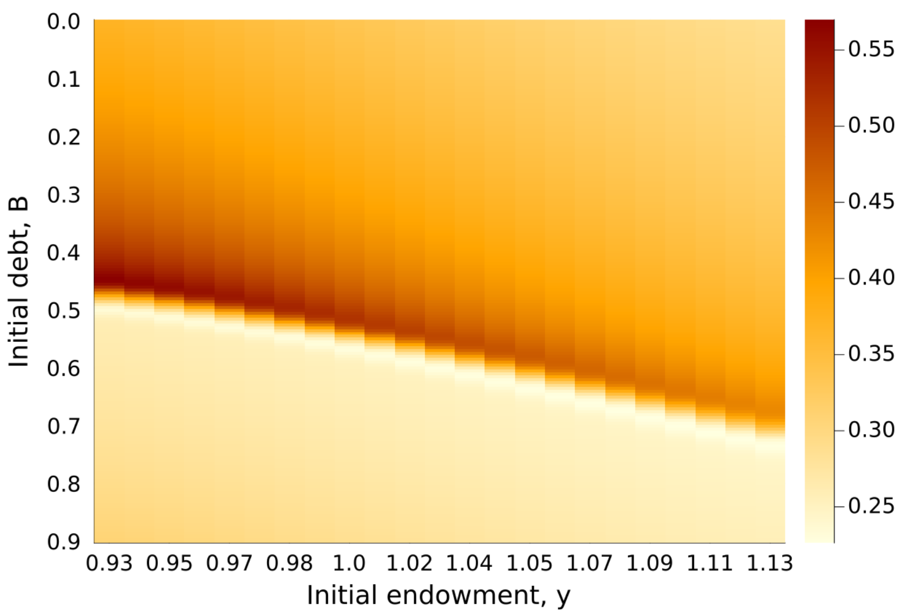


Figure 15: Equivalent variation $\zeta(y_0, B_0)$

Figure 15 shows a heat map for the equivalent variation, ζ, in percentage terms, for different levels of y_0 and B_0.

Noticeably, the equivalent variation is strictly positive for every initial state (y_0, B_o), highlighting that, under the calibrated model, the uniform price protocol is preferred to the discriminatory price protocol. A closer inspection of the figure provides further insight. At the top right corner, with high endowment and zero debt, the difference between protocols is at the lower end of the interval. However, once we start moving down the initial endowment and increasing the level of initial debt, towards the bottom left corner, the difference starts increasing. This pattern has to do with the increase in the likelihood of default that follows from a decrease in endowment coupled with an increase in debt. As default becomes more likely the difference in the protocols increases, up to the point where default is certain. One could infer this through the differences in the bid schedules. When default is extremely unlikely, the bid schedules are closer together, however, as the likelihood of default increases, static dilution is more noticeable and, as a result, dynamic dilution also becomes more pronounced under the discriminatory price protocol. The access to the insurance benefits of the discriminatory price protocol are too costly.

7.2 Lender's Welfare

We next discuss how we compare lender's welfare. Consistently with how we have proceeded with government's welfare, we define lender's welfare as the beginning of period value to the lender of holding a bond. That is,

$$Q_{ante}(s_0, B) = \mathbb{E}_{T,m,s'}\left[(1-d)\big((\kappa+\lambda)+(1-\lambda)Q(s', B')\big) + dQ^D(s_0, B)|s_0\right]$$

where d, B' are policy rules and $Q(\cdot)$ and $Q^D(\cdot)$ are as described before.

Since lenders are risk neutral, the values are already in units of consumption. We compute the equivalent variation in consumption as the difference in $Q_{ante}(\cdot)$ across protocols:

$$\zeta_L(s_0, B_0) = Q^{UP}_{ante}(s_0, B) - Q^{DP}_{ante}(s_0, B)$$

where Q^{UP}_{ante} and Q^{DP}_{ante} denote the ex-ante value under the uniform and discriminatory price protocols, respectively.

Figure 16 shows a heat map for the equivalent variation, ζ_L, for different levels of y_0 and B_0. As expected, differences in prices are very close to zero absent default risk. These differences became meaningful as the country nears default. As the likelihood of default increases, static dilution within an auction, and the corresponding effect on dynamic dilution, get more pronounced, leading to lower prices under the discriminatory price protocol. Noticeably, the equivalent variation, ζ_L, is non-negative. As such, the value to the lenders is also larger under the uniform price protocol.

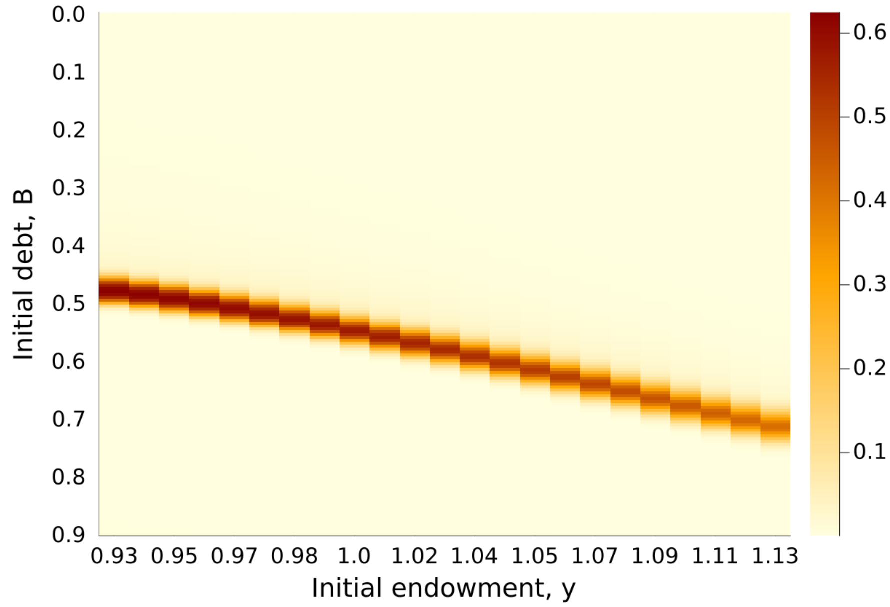


Figure 16: Equivalent variation $\zeta_L(s_0, B_0)$

7.3 Discussion

It turns out that, under the model calibrated to the Portuguese economy, the insurance component of the discriminatory protocol is more than offset by the dilution effects. In fact, the uniform price protocol protects investors from being diluted within an auction, while at the same time provides better incentives on government's borrowing over time. The result that the insurance component is more than offset is consistent with the well known fact that the welfare costs of fluctuations are small, as in Lucas (1987), and as such the benefits of insurance are limited for aggregate shocks. Alternatively, one can say that access to insurance under the discriminatory price protocol is too costly. Investors internalize that the government will use the insurance, through overborrowing, and because of that offer lower prices.

The world under a uniform price protocol is better than the world under a discriminatory price protocol. Not only is government's welfare higher, but so is lender's welfare. The increase in government's welfare does not come at the expense of lenders, instead, using the uniform price protocol when default risk is a concern is a Pareto improvement.

The result of this counterfactual exercise is consistent with the switch in protocol observed in the data: Portugal switched from a discriminatory price protocol to a uniform price protocol. In particular, Portugal stopped issuing securities with maturity longer than one year during from 2011 to 2014 and the switch occurred right before the return of the Portuguese Treasury to financial markets, for those same maturities.

It is possible to reconcile the timing of the change observed in the data with the results depicted in the heat maps above. If we assume that changing the protocol of the auctions involves switching costs, then the government would wait for a state such that the gain from switching is greater than the costs incurred by doing so. The gains are larger when the government is faced with a combination of high debt and low endowment, after a recession hits. This is consistent with switching protocols during the crisis. Moreover, the difference in lender's welfare across protocols also fits with the switching costs argument[24] reconciling the timing of the switch in protocols.

What he have not explained is why would Portugal use a discriminatory price protocol in the first place. To that effect, we must highlight that the results shown here are in an environment in which default risk is a concern. As such, we do not model an environment where default risk is not a driving force governing the value of debt. One could argue that prior to the 2010s, it was not expected that Europe would have a sovereign debt crisis as this type of event was typically associated with emerging markets. This explanation is consistent with the flat bid schedules that we observe in the data for the early 2000s (recall

[24] If we were to consider that lenders could influence the government's choice of which protocol to use.

Figure 3). In such an environment, with little to no uncertainty, the two protocols are close to equivalent and using the discriminatory price protocol does not seem as outlandish as when default risk is a concern.

8 Conclusion

In this paper, we compared the two most widely used auction protocols to issue sovereign debt, when default risk is a concern. Letting the government choose how much to borrow after observing the demand for debt, we took into account how different protocols affect not only investors' but also government's decisions over time.

We built a theoretical model of sovereign borrowing and default with repeated auctions and asymmetric information on government's public spending. We disciplined the model with proprietary bid level data for Portuguese sovereign debt auctions, data on differences between realized and expected public spending, as well as institutional details relevant for modeling sovereign debt issuances. We then validated the model calibrated to the Portuguese economy. The calibrated model under the discriminatory price protocol, was capable of matching standard moments in the Portuguese economy regarding debt, spreads and business cycles statistic. Furthermore, the use of a discriminatory protocol enabled the model to easily generate spreads whose volatility significantly exceeds their mean, a shortcoming of previous sovereign debt models. In a counterfactual exercise, we compared the two auction protocols. We found that the benefits of switching from a discriminatory to a uniform price protocol are increasing in the likelihood of default and go up to 0.6% of permanent consumption. Moreover, switching to a uniform price protocol is a Pareto improvement as both the small open economy and foreign lenders are better off after the switch. This result is consistent with the change in protocol observed in the data: Portugal switched from a discriminatory to a uniform price protocol in the aftermath of the sovereign debt crisis.

Finally, we also found that dynamics, through repeated auctions, are key in separating the outcomes under the two protocols. Even though for reasonable parameter values the discriminatory price protocol performs better than the uniform under a single auction setting, in the model calibrated to the Portuguese economy, the uniform price protocol is preferred. In fact, when default risk is a concern, the uniform price protocol protects investors from static dilution within an auction and at the same time provides better incentives on borrowing over time.

With this framework we were able to rationalize the switch of auction protocol in Portugal, as well as the timing of the switch. The main source of risk affecting the price of sovereign debt in our model is default risk. As such, we do not attempt to explain why Portugal was using a discriminatory price protocol prior to the crisis. As we have argued, prior to the 2010s, it was not expected that Europe would have a sovereign debt crisis as this type of event was typically associated with emerging markets. In such an environment, with little to no uncertainty, the two protocols are close to equivalent and using the discriminatory price protocol does not seem as outlandish as when default risk is a concern.

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Appendix

Appendix A: Full Definition of Equilibrium

Given the auction protocol, a stationary symmetric recursive equilibrium consists of:

1. Value functions V, V^R, V^D;

2. Price functions Q, Q^D;

3. Bid price function p;

4. Policy functions B', ℓ, P_c, d.

such that the following conditions are satisfied:

1. Default decision optimality: given V^R and V^D, d solves the government's default or repayment decision and V is the resulting value function;

2. Borrowing decision optimality: given V and p, $\{B', P_c, \ell\}$ solve the government's repayment problem and V^R is the resulting value function;

3. Asset pricing in good standing: given d, B' and Q^D, Q satisfies the functional equation defining the value of debt while in good standing;

4. Value of default: given V, V^D is the value function for the government upon default;

5. Asset pricing in default: given Q, Q^D satisfies the functional equation defining the value of a defaulted bond;

6. Bid optimality: given Q, B' and P_c, p satisfies the bid optimality condition of ex-ante zero profits for investors;

7. Auction clearing: given p, P_c and B', the sum of accepted bid quantities equals the debt issuance, $\ell \equiv B' - (1 - \lambda)B$.

Appendix B: Omitted Proofs

Theorem 1 (Revenue Equivalence). *If B' is a random variable independent of the auction protocol, then ex-ante expected revenue in the auction is the same under both protocols.*

Proof:

Let B' be a continuous random variable with cdf G. Let, as before, Δ^D and Δ^U denote, respectively, revenue under the discriminatory and uniform price protocols.

$$\begin{aligned}
\mathbb{E}[\Delta^D(b')] &= \int_0^{b_H} \left[\int_0^{b'} p(b)db\right] dG(b') \\
&= \int_0^{b_H} p(b) \left[\int_b^{b_H} dG(b')\right] db \\
&= \int_0^{b_H} p(b)[1 - G(b)]db \\
&= R^{-1} \int_0^{b_H} \left[\int_b^{b_H} F(\underline{v}^d(\tilde{b}))dG(\tilde{b})\right] db \\
&= R^{-1} \int_0^{b_H} F(\underline{v}^d(\tilde{b}))g(\tilde{b}) \left[\int_0^{\tilde{b}} db\right] d\tilde{b} \\
&= R^{-1} \int_0^{b_H} \tilde{b}F(\underline{v}^d(\tilde{b}))dG(\tilde{b}) \\
&= \mathbb{E}[\Delta^U(b')] \qquad \square
\end{aligned}$$

The first equality defines expected revenue under a discriminatory price protocol. For the second equality we proceed by changing the order of integration. After simplifying the expression, and substituting $p(b)$ by its equilibrium expression, for the fifth equality we perform another change in the order of integration. Simplifying yields the definition of expected revenue under the uniform price protocol.

Proposition 1. *Bidding marginal prices is a dominant strategy for investors.*

Proof: Let $\mathcal{P}$ denote the set of marginal prices chosen by the government. A marginal price $P_c(\theta; p)$, depends on the realization of θ, for a given aggregate bid schedule. As θ is drawn from a discrete distribution with finite support, the set $\mathcal{P}$ is itself finite.

Let the shock θ take two possible values, further, let the associated marginal prices be $P_{c,1} > P_{c,2}$. Consider a bid with price p. Suppose for the sake of contradiction that a bid function (p_1, p_2) that has $p_1 = P_{c,1} > p_2 > P_{c,2}$ dominates (p_1, p_2') that has $p_1 = P_{c,1} > p_2' = P_{c,2}$.

The first bid is the same across bid functions so we can focus on the second bid. First note that the bid with price p_2 is accepted with the same probability of a bid with price $p_2' = P_{c,2}$, as there is no marginal price chosen in between $P_{c,1}$ and $P_{c,2}$. It follows that

$$Pr[P_c(\theta) \leq p_2] = Pr[P_c(\theta) \leq p_2'] = Pr[P_c(\theta) \leq P_{c,2}]$$

Second, the value of debt only depends on the marginal price, that is not affected by an individual infinitesimal dealer. In particular we have $Q(\ell(P_{c,1})) \geq Q(\ell(P_{c,2}))$. That is, bidding p_2 instead of p_2' does not affect the value of debt, the investors payoff if the bid is executed. The cost associated with bidding $p_2 > P_{c,2}$, however, is greater than bidding $p_2' = P_{c,2}$. The dealer's unitary profit for this bid is then:

$$Pr[P_c(\theta) \leq P_{c,2}]\Big(Q(\ell(P_{c,2})) - \phi(p_2, P_c(\theta)|\ell(P_{c,2}))\Big)$$

For a discriminatory price protocol $\phi(p_2, \cdot) = p_2$ and

$$Pr[P_c(\theta) \leq P_{c,2}]\Big(Q(\ell(P_{c,2})) - p_2\Big) < Pr[P_c(\theta) \leq P_{c,2}]\Big(Q(\ell(P_{c,2})) - p_2'\Big) \tag{5}$$

Let the uniform price protocol be the limiting case when $\epsilon \to 0$ as follows: $\epsilon p_2 + (1 - \epsilon)P_{c,2}$, then in the limit,

$$\epsilon p_2 + (1-\epsilon)P_{c,2} - (\epsilon p_2' + (1-\epsilon)P_{c,2}) \to 0 \tag{6}$$

and the government is indifferent between bidding p_2 and p_2'.

Note that (5) and (6) contradict $p = (p_1, p_2)$ dominating $p' = (p_1, p_2')$. It follows that bidding marginal prices is a strictly dominant strategy under a discriminatory price protocol and a weakly dominant strategy under a uniform price protocol. $\square$

Appendix C: Two Period Environment, Alternative Specification

Consider the environment described in section 4. Instead of unexpected spending as a random variable, consider a preference shock in the first period. In particular, preferences over streams of consumption are as follows:

$$\mathbb{E}\left[\theta u(c_0) + \beta u(c_1)\right]$$

The taste shock, θ, is privately observed by the government. It is drawn from a continuous distribution with support on $[\theta_L, \theta_H]$ with $\theta_L < \theta_H$ and cdf G. We further assume that $g(\theta) = G'(\theta) > 0$ on $[\theta_L, \theta_H]$.

We use the same parameterization as before with one difference. Suppose that $v^d = y(1 - \exp(-z))$ with z distributed exponentially with cdf $F(z) = 1 - exp(-\mu z)$ and $z = -ln(1 - v^d/y)$. Then $F(v^d) = 1 - exp(\mu ln(1 - v^d/y)) = 1 - (1 - v^d/y)^{\mu}$ and $F'(v^d) = (\mu/y)(1 - v^d/y)^{\mu-1}$. For $\mu = 1$ this collapses into an uniform distribution on the interval $[0, y]$.

Commitment to a Borrowing Rule

We first tie the hands of the government. Suppose the government could commit to a borrowing rule, θ is observed ex-post and the government commits to it. In particular, the government commits to the optimal borrowing rule under UP, regardless of the protocol being used. That is, $b(\theta) = b(\theta)_{UP}$. By fixing the distribution of b' across protocols, utility in the second period is independent of the protocol. Furthermore, we recover revenue equivalence. With linear utility, welfare is pinned down by

$$\mathbb{E}[\theta\Delta(\theta)] = \mathbb{E}[\theta]\mathbb{E}[\Delta(\theta)] + \mathbb{C}(\theta, \Delta(\theta))$$

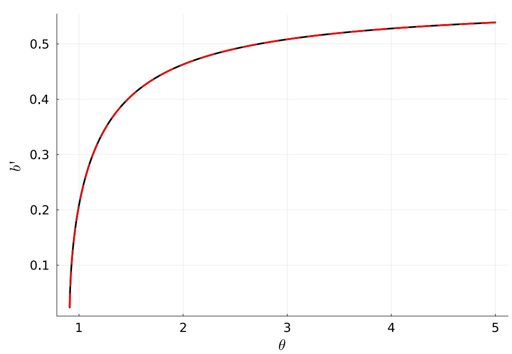
In particular, the difference in welfare across protocols is determined by the covariance term. This term is the insurance component that stems from the curvature introduced by the multiplicative taste shock, θ.

In Figure 17 below we can see how the protocols compare. Panel (a) illustrates the commitment to a borrowing rule as a function of θ. Panel (b) shows us that static dilution is still present, with the bid schedule under DP lower than the one under UP. Panel (c) highlights the potential benefits of insurance, higher revenue in bad states at the expense of relative lower revenue in good states. Panel (d) shows us that welfare tends to be higher under DP, particularly when there are large financing needs in the first period.

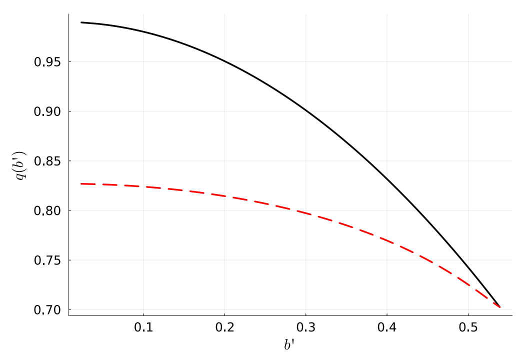
Ex-ante welfare is higher under DP:

$$\mathbb{E}[V(\theta)_{UP}] = 1.754 \quad \mathbb{E}[V(\theta)_{DP}] = 1.761$$

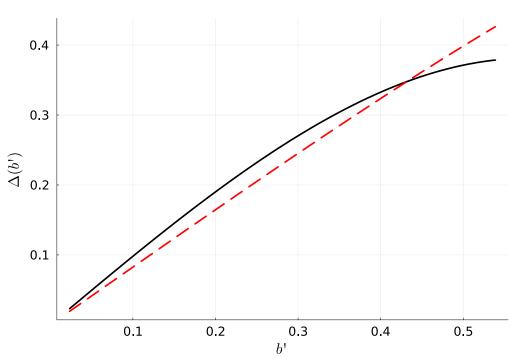
It follows that the covariance term is larger under DP.



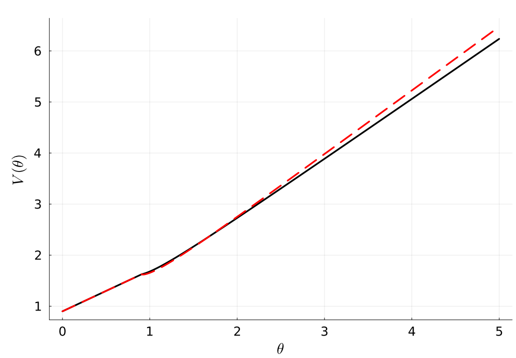
(a) Borrowing Decisions



(b) Bid Schedules



(c) Revenue



(d) Value functions

Figure 17: Comparing outcomes under commitment to $b(\theta)_{UP}$

To evaluate the cost of not committing to the borrowing rule, we let the government choose optimally for each realization of θ, given the price schedule. Note that welfare under UP will be unchanged as the government was already choosing optimally. As such, the difference in welfare under DP measures the static dilution that arises from the lack of commitment. This specification of the environment allows us to get a closed form solution detailed below.

Closed Form Solution

The equilibrium under a uniform price protocol is fairly standard to solve. Under the specified functional forms, we can also find a closed-form solution for the fixed point problem described above, between investors' and government's strategies, under a discriminatory price protocol.

Consider linear preferences, such that $u(x) = x$. Let us first characterize the equilibrium under a discriminatory price protocol. An equilibrium requires an actuarially fair price for investors and attaining the maximum in problem (1), respectively:

$$\begin{aligned} p(b) &= \frac{1}{1 - G(\theta(b))} \int_{\theta(b)}^{\theta_H} Q(b(\theta)) dG(\theta) \\ \theta p(b(\theta)) &= \beta F(y - b(\theta)) \end{aligned}$$

These two conditions together give us a single optimality condition:

$$\frac{\theta}{1 - G(\theta)} \int_{\theta}^{\theta_H} F(y - b(x)) dG(x) = \beta R F(y - b(\theta))$$

Before we solve the equation above, let us just point out that the solution relies on the fact that for a small enough θ the government does not borrow, $b(\theta) = 0$, and so the probability of repayment equals one. As $\theta \to 0$ the benefit of borrowing goes to zero as U only depends on consumption in the second period. Below we show that borrowing is non-decreasing in θ.

Proposition 2 (Monotonicity of b)**.** *If u is strictly increasing and concave, $\beta \in (0, 1)$, and*

$f(v^d) = F'(v^d) > 0$ on $[\underline{v}, \overline{v}]$, then $b(\theta; p)$ is non-decreasing in θ.

Proof: Suppose, for the sake of contradiction that $b(\theta)$ is strictly decreasing in θ.

A government chooses b such that:

$$\theta u'(y + \Delta(b) - b_0)\Delta'(b) = \beta F(u(y - b))u'(y - b) \tag{7}$$

Let $\theta_2 > \theta_1 > 0$ and $b(\theta_1)$ and $b(\theta_2)$ be the optimal choices associated with θ_1 and θ_2, respectively. Note that the marginal benefit of borrowing (left-hand side) is non-increasing in b. First, $u'(y + \Delta(b) - b_0)$ is non-increasing in $\Delta(b)$ as u is concave by assumption; $\Delta(b)$ is a concave function of b as the price schedule is a non-increasing function of b and so $\Delta'(b)$ is also non-increasing. Further, optimality requires that $\Delta'(b) \geq 0$ along the equilibrium path. As $\theta_2 > \theta_1$ and $b(\theta_1) > b(\theta_2)$, it follows that:

$$\theta_2 u'(y + \Delta(b(\theta_2)) - b_0)\Delta'(b(\theta_2)) > \theta_1 u'(y + \Delta(b(\theta_1)) - b_0)\Delta'(b(\theta_1)) \tag{8}$$

Pick a value of y (or $\overline{v}$) such that $F(u(y - b(\theta_2))) = 1$, that is, $u(y - b(\theta_2)) > u(y - b(\theta_1)) \geq \overline{v}$. Then, for $\theta \in \{\theta_1, \theta_2\}$ the government never defaults and $Q(b(\cdot)) = R^{-1}$.

The marginal cost of borrowing, when default is a zero probability event, is non-decreasing in b, as u is concave and $F(\cdot)$ is constant and equal to 1. As $b(\theta_1) > b(\theta_2)$, it follows that:

$$\beta F(u(y - b(\theta_2)))u'(y - b(\theta_2)) \leq \beta F(u(y - b(\theta_1)))u'(y - b(\theta_1))$$

Note that equation (7) then requires:

$$\theta_2 u'(y + \Delta(b(\theta_2)) - b_0)\Delta'(b(\theta_2)) \leq \theta_1 u'(y + \Delta(b(\theta_1)) - b_0)\Delta'(b(\theta_1))$$

which contradicts equation (8). $\square$

Under a discriminatory price protocol, we have established that, with linear preferences, for any θ that has the government borrowing in equilibrium, it must be that:

$$\theta \frac{R^{-1}}{1-G(\theta)} \int_{\theta}^{\theta_H} F(\underline{v}^d(b(x))dG(x) = \beta F(y - b(\theta))$$

Let $n(\theta) \equiv F(y - b(\theta))$, denote the probability of repayment at θ. Then, the equation above is

$$\theta \int_{\theta}^{\theta_H} n(x) \frac{dG(x)}{1-G(\theta)} = \beta R \, n(\theta)$$

Set $N(\theta) \equiv \int_{\theta}^{\theta_H} n(x)dG(x)$ and $N'(\theta) = -n(\theta)dG(\theta)$. Then,

$$\theta N(\theta) = -\beta R \frac{1-G(\theta)}{dG(\theta)} N'(\theta) \iff \frac{N'(\theta)}{N(\theta)} = -\frac{\theta}{\beta R} \frac{dG(\theta)}{1-G(\theta)}$$

For an exponentially distributed θ, we have $G(x) = 1 - exp(-\lambda x)$ and $dG(x) = \lambda exp(-\lambda x)$.

$$\frac{N'(\theta)}{N(\theta)} = -\frac{\theta}{\beta R} \frac{1-G(\theta)}{dG(\theta)} \iff \log(N(\theta)) = -\frac{\theta^2 \lambda}{2\beta R} + C \implies N(\theta) = K.exp\left(-\frac{\theta^2\lambda}{2\beta R}\right)$$

where $K = exp(C)$. Taking derivatives we get:

$$N'(\theta) = -K \frac{\theta}{\beta R} exp\left(-\frac{\theta^2\lambda}{2\beta R}\right)$$

Recalling that $N'(\theta) = -n(\theta)dG(\theta)$, by definition this is equivalent to:

$$n(\theta) = K \frac{\theta}{\beta R} exp\left(\lambda\theta - \frac{\theta^2\lambda}{2\beta R}\right)$$

This must be true for some $K > 0$. To determine the value of K, we make some assumptions about the nature of the equilibrium. In general, an equilibrium of the kind we posit always exists. We look for equilibria in which 1) there is a $\hat{\theta}$, such that the government's first order condition holds at $b' = 0$ (and therefore $n(\hat{\theta}) = 1$), and 2) at this $\hat{\theta}$, it is the case that $n'(\hat{\theta}) = 0$. The second condition selects a specific θ. In particular, it selects the lowest possible one. We begin solving for $\hat{\theta}$ and K by examining the implications of $n'(\hat{\theta}) = 0$.

The derivative of $n(\theta)$ is:

$$n'(\theta) = K\frac{1}{\beta R}exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right) + K\frac{\theta}{\beta R}\left(\lambda - \frac{\lambda\theta}{\beta R}\right)exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right)$$

Collect terms to rewrite this as:

$$n'(\theta) = K\frac{1}{\beta R}\left(1 + \lambda\theta\left(1 - \frac{\theta}{\beta R}\right)\right)exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right)$$

Since the collection of terms outside the big parentheses are all positive, we see that this is a parabola that opens down. Setting it equal to 0 yields:

$$1 + \lambda\theta - \frac{\lambda\theta^2}{\beta R} = 0$$

We will want the right root of this (so that $n'(\theta)$ is appropriately negative for $\theta \geq \hat{\theta}$). The above can be rewritten as:

$$\theta^2 - \beta R\theta - \frac{\beta R}{\lambda} = 0$$

Then $\hat{\theta}$ is given by:

$$\hat{\theta} = \frac{\beta R + \sqrt{(\beta R)^2 + 4\frac{\beta R}{\lambda}}}{2}$$

Finally, having solved for $\hat{\theta}$ in terms of parameters, we can quickly solve for K as the solution to:

$$1 = n(\hat{\theta}) = K\frac{\hat{\theta}}{\beta R}exp\left(\lambda\hat{\theta} - \frac{\lambda\hat{\theta}^2}{2\beta R}\right)$$

So:

$$K = \frac{\beta R}{\hat{\theta}}exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right)$$

Then, $n(\theta)$ becomes:

$$n(\theta) = \frac{\beta R}{\hat{\theta}}exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right)\frac{\theta}{\beta R}exp\left(\lambda\theta - \frac{\theta^2\lambda}{2\beta R}\right)$$

which can be simplified to:

$$
\begin{aligned}
n(\theta) &= \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta-\hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2-\hat{\theta}^2)\right) \\
&= \frac{\theta}{\hat{\theta}}\, exp\left(-\frac{\lambda}{\beta R}(\theta-\hat{\theta})\left(\frac{\theta+\hat{\theta}}{2} - \beta R\right)\right)
\end{aligned}
$$

Given a functional form of $F(\cdot)$, this can then be mapped back to choices of b' using the definition:

$$n(\theta) = F(y - b'(\theta))$$

Suppose that $v^d = y(1-\exp(-z))$ where z is distributed exponentially with cdf $F(z) = 1 - exp(-\mu z)$ and $z = -ln(1-v^d/y)$. Then $F(v^d) = 1 - exp(\mu ln(1-v^d/y)) = 1 - (1-v^d/y)^{\mu}$ and $F'(v^d) = (\mu/y)(1-v^d/y)^{\mu-1}$. When $v^d = y - b(\theta)$, the term $1 - v^d/y$ becomes:

$$1 - \frac{y-b(\theta)}{y} = \frac{y-y+b(\theta)}{y} = \frac{b(\theta)}{y}$$

Using the optimality condition (8) we get:

$$
\begin{aligned}
& \beta F(y-b(\theta)) = \frac{\theta R^{-1}}{exp(-\lambda\theta)} N(\theta) \iff \\
\iff & \beta\left(1 - \left(\frac{b(\theta)}{y}\right)^{\mu}\right) = \frac{\theta R^{-1}}{exp(-\lambda\theta)} \frac{\beta R}{\hat{\theta}} exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right) exp\left(-\frac{\theta^2\lambda}{2\beta R}\right) \\
\iff & 1 - \left(\frac{b(\theta)}{y}\right)^{\mu} = \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta-\hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2-\hat{\theta}^2)\right) \\
\iff & b(\theta) = y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta-\hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2-\hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}
\end{aligned}
$$

Recall that $p(b(\theta)) = \frac{\beta}{\theta}F(y - b(\theta))$, it then follows that:

$$\begin{aligned}
p(b(\theta)) &= \frac{\beta}{\theta}\left(1 - \left(\frac{b(\theta)}{y}\right)^{\mu}\right) \\
&= \frac{\beta}{\theta}\left(1 - \left[\frac{y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}}{y}\right]^{\mu}\right) \\
&= \frac{\beta}{\theta}\left(\frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right) \\
&= \frac{\beta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right) \\
&= p(\theta)
\end{aligned}$$

Under a uniform price protocol, an equilibrium with positive borrowing requires:

$$\begin{aligned}
&\theta \Delta'(b(\theta)) = \beta F(y - b(\theta)) \iff \\
\iff &\theta\left[Q(b(\theta)) + \frac{\partial Q(b(\theta))}{db(\theta)} b(\theta)\right] = \beta F(y - b(\theta)) \\
\iff &\theta\left[R^{-1}F(y - b(\theta)) + R^{-1}F'(y - b(\theta))b(\theta)\right] = \beta F(y - b(\theta)) \\
\iff &\theta\left[1 - \frac{F'(y - b(\theta))}{F(y - b(\theta))} b(\theta)\right] = \beta R \\
\iff &\theta \frac{F'(y - b(\theta))}{F(y - b(\theta))} b(\theta) = \theta - \beta R \\
\iff &\theta \frac{\frac{\mu}{y}\left(\frac{b(\theta)}{y}\right)^{\mu - 1}}{1 - \left(\frac{b(\theta)}{y}\right)^{\mu}} b(\theta) = \theta - \beta R \\
\iff &\theta \frac{\mu}{y^{\mu}} b(\theta)^{\mu} = (\theta - \beta R) - (\theta - \beta R) b(\theta)^{\mu} \frac{1}{y^{\mu}} \\
\iff &b(\theta)^{\mu}\left(\theta \frac{\mu}{y^{\mu}} + \theta \frac{1}{y^{\mu}} - \beta R \frac{1}{y^{\mu}}\right) = \theta - \beta R \\
\iff &b(\theta) = \left(\frac{\theta - \beta R}{\theta(1 + \mu) - \beta R}\right)^{\frac{1}{\mu}} y
\end{aligned}$$

We have established that under a uniform price protocol investors only bid marginal prices, hence:

$$
\begin{aligned}
p(b(\theta)) &= R^{-1}F(y-b(\theta)) \\
&= R^{-1}\left(1-\left(\frac{\theta-\beta R}{\theta(1+\mu)-\beta R}\right)\right) \\
&= R^{-1}\left(\frac{\mu\theta}{\theta(1+\mu)-\beta R}\right) \\
&= p(\theta)
\end{aligned}
$$

Summing up, for a **uniform price auction**:

$$
b(\theta) = \left(\frac{\theta-\beta R}{\theta(1+\mu)-\beta R}\right)^{\frac{1}{\mu}} y
$$

$$
p(\theta) = R^{-1}\left(\frac{\mu\theta}{\theta(1+\mu)-\beta R}\right)
$$

And for a **discriminatory price auction**:

$$
b(\theta) = y\left(1-\frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta-\hat{\theta})-\frac{\lambda}{2\beta R}(\theta^2-\hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}
$$

$$
p(\theta) = \frac{\beta}{\hat{\theta}}\, exp\left(\lambda(\theta-\hat{\theta})-\frac{\lambda}{2\beta R}(\theta^2-\hat{\theta}^2)\right)
$$

Appendix D: Robustness

Utility

Let us first see what happens under different utility functions. All other parameters are the same as before.

$$\begin{aligned}
\text{Linear Utility}: \ & \mathbb{E}[V(\theta)_{UP}] = 1.754 > \mathbb{E}[V(\theta)_{DP}] = 1.698 \\
\text{Log Utility}: \ & \mathbb{E}[V(\theta)_{UP}] = -0.0986 < \mathbb{E}[V(\theta)_{DP}] = -0.0973 \\
\text{CRRA}, \gamma = 2: \ & \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
\text{CRRA}, \gamma = 4: \ & \mathbb{E}[V(\theta)_{UP}] = -0.8065 < \mathbb{E}[V(\theta)_{DP}] = -0.8059 \\
\text{CRRA}, \gamma = 8: \ & \mathbb{E}[V(\theta)_{UP}] = -0.5146 < \mathbb{E}[V(\theta)_{DP}] = -0.5137
\end{aligned}$$

Distribution of θ

Let us keep CRRA with $\gamma = 2$ and v^d uniformly distributed on $[\underline{v}, \overline{v}]$.

$$\begin{aligned}
\theta \sim Exp(1): \ & \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
\theta \sim U(0,5): \ & \mathbb{E}[V(\theta)_{UP}] = -3.5773 < \mathbb{E}[V(\theta)_{DP}] = -3.5762 \\
\theta \sim U(0,10): \ & \mathbb{E}[V(\theta)_{UP}] = -5.6877 < \mathbb{E}[V(\theta)_{DP}] = -5.6851 \\
\theta \sim N(3,2): \ & \mathbb{E}[V(\theta)_{UP}] = -4.1615 < \mathbb{E}[V(\theta)_{DP}] = -4.1599
\end{aligned}$$

Distribution of v^d

Let us keep CRRA with $\gamma = 2$ and θ exponentially distributed with $\lambda = 1$.

$$\begin{aligned}
v^d \sim U(\underline{v}, \overline{v}): \ & \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
v^d \sim N(u(0.2) = -5, 1.5): \ & \mathbb{E}[V(\theta)_{UP}] = -2.0254 < \mathbb{E}[V(\theta)_{DP}] = -2.0248
\end{aligned}$$

Output Growth

Let us keep CRRA with $\gamma = 2$, θ exponentially distributed with $\lambda = 1$ and v^d uniformly distributed.

$$y_1 = y_0: \ \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259$$
$$y_1 = 1.05 \times y_0: \ \mathbb{E}[V(\theta)_{UP}] = -1.9677 < \mathbb{E}[V(\theta)_{DP}] = -1.9671$$
$$y_1 = 0.95 \times y_0: \ \mathbb{E}[V(\theta)_{UP}] = -2.0896 < \mathbb{E}[V(\theta)_{DP}] = -2.0891$$

Budget Deficits

Instead of considering a multiplicative taste shock we now look at what would happen if instead uncertainty is regarding a budget deficit, θ as follows:

$$c = y + \Delta(b(\theta)) - b_0 - \theta$$

Let us keep CRRA with $\gamma = 2$ and v^d uniformly distributed. θ is exponentially distributed with $\lambda = 1$ and truncated to the interval $[0, 1]$.

$$\mathbb{E}[V(\theta)_{UP}] = -2.8976 < \mathbb{E}[V(\theta)_{DP}] = -2.8952$$

Figure 15 shows a heat map for the equivalent variation, ζ, in percentage terms, for different levels of y_0 and B_0.

Appendix E: Computational Details

The set of objects used to solve the model numerically and assess convergence are as follows:

1. The continuation value functions $W(s, T, B, B')$ and $W^D(s, T, B)$, given by:

$$W(s, T, B, B') = \mathbb{E}[V(s', T', \mathbf{m}, B, B')|s]$$
$$W^D(s, T, B) = \mathbb{E}[V^D(s', T', \mathbf{m}, B)|s]$$

2. The price functions $Q(s, B')$ and $Q^D(s, B)$ and the expected probability of default $\delta(s, B')$.

In short, these are the continuation value functions, the price functions, and the expected probability of default. Note that there are other price and value functions (including the bid function in the discriminatory price protocol), but they can be derived based on the above set of objects and within-period optimization. We use the above set as the list to assess convergence.

These objects are defined on grids of their arguments. In particular, we have the following sets that we will need to define grids for:

1. $s \in \mathcal{S}$, that defines GDP and expected public spending.
   (a) For the grid of GDP values, $y(s)$, we use 23 points evenly spaced in logs spread across a space spanning six of the logged variable's long run standard deviations and centered at its mean:

$$[\mathbb{E}[\log(y(s))] - 3\sigma[\log(y(s))], \mathbb{E}[\log(y(s))] + 3\sigma[\log(y(s))]]$$

   (b) For the grid of expected public spending values, $g(s)$, we use 17 points evenly spaced in logs spread across a space spanning six of the logged variable's long run standard deviations and centered at its mean:

$$[\mathbb{E}[\log(g(s))] - 3\sigma[\log(g(s))], \mathbb{E}[\log(g(s))] + 3\sigma[\log(g(s))]]$$

2. $B \in \mathcal{B}$: for the grid of b we use 241 evenly spaced points on [0, 1.2].
3. $T \in \mathcal{T}$, that defines surprise budget spending: for the grid of $\theta(T)$ we use 31 points evenly spaced spanning six of the logged variable's long run standard deviations and centered at one (the average log is zero).

Given a guess for the set of objects listed above, in order to generate a new guess, the iteration proceeds as follows:

1. Using the baseline set of objects, and given the restructuring structure upon regaining access to financial markets, generate new guesses for $W^D(s, T, B)$ and $Q^D(s, B)$.
2. Using the baseline set of objects, and those defined in the previous step, solve the government's problem when it enters a period in good standing. Use the solution to generate new guesses of $W(s, T, B, B')$, $Q(s, B')$ and $\delta(s, B')$.
3. Check the sup-norm distance between all objects. If it is less than 10^{-5}, stop. Otherwise, update guesses using rules of the form

$$f_{next}(\cdot) = \xi_j f_{old}(\cdot) + (1 - \xi_j) f_{new}(\cdot)$$

   where $j \in \{V, Q\}$, and return to step 1.

   This type of rule updates the old guess by moving fraction $(1 - \xi_j)$ of the distance towards the new guess. In general, to ensure convergence, updates of the the price functions tend to require more smoothing than those of the value functions. Moreover, solving the government's problem in good standing under a discriminatory price protocol also requires smoothing for the update of bid schedules and auction revenue.