Online Appendix for Sovereign Debt Auctions with Strategic Interactions *

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Appendix A: Full Definition of Equilibrium

Given the auction protocol, a stationary symmetric recursive equilibrium consists of:

1. Value functions V, V^R, V^D;
2. Price functions Q, Q^D;
3. Bid price function p;
4. Policy functions B', ℓ, P_c, d.

such that the following conditions are satisfied:

1. Default decision optimality: given V^R and V^D, d solves the government's default or repayment decision and V is the resulting value function;
2. Borrowing decision optimality: given V and p, $\{B', P_c, \ell\}$ solve the government's repayment problem and V^R is the resulting value function;
3. Asset pricing in good standing: given d, B' and Q^D, Q satisfies the functional equation defining the value of debt while in good standing;
4. Value of default: given V, V^D is the value function for the government upon default;

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5. Asset pricing in default: given Q, Q^D satisfies the functional equation defining the value of a defaulted bond;

6. Bid optimality: given Q, B' and P_c, p satisfies the bid optimality condition of ex-ante zero profits for investors;

7. Auction clearing: given p, P_c and B', the sum of accepted bid quantities equals the debt issuance, $\ell \equiv B' - (1-\lambda)B$.

Appendix B: Omitted Proofs

Theorem 1 (Revenue Equivalence). *If B' is a random variable independent of the auction protocol, then ex-ante expected revenue in the auction is the same under both protocols.*

Proof: Let B' be a continuous random variable with cdf G. Let, as before, Δ^D and Δ^U denote, respectively, revenue under the discriminatory and uniform price protocols.

$$
\begin{aligned}
\mathbb{E}[\Delta^D(b')] &= \int_0^{b_H} \left[\int_0^{b'} p(b)db \right] dG(b') \\
&= \int_0^{b_H} p(b) \left[\int_b^{b_H} dG(b') \right] db \\
&= \int_0^{b_H} p(b)[1 - G(b)]db \\
&= R^{-1} \int_0^{b_H} \left[\int_b^{b_H} F(\underline{v}^d(\tilde{b}))dG(\tilde{b}) \right] db \\
&= R^{-1} \int_0^{b_H} F(\underline{v}^d(\tilde{b}))g(\tilde{b}) \left[\int_0^{\tilde{b}} db \right] d\tilde{b} \\
&= R^{-1} \int_0^{b_H} \tilde{b}F(\underline{v}^d(\tilde{b}))dG(\tilde{b}) \\
&= \mathbb{E}[\Delta^U(b')] \qquad \square
\end{aligned}
$$

The first equality defines expected revenue under a discriminatory price protocol. For the second equality we proceed by changing the order of integration. After simplifying the expression, and substituting $p(b)$ by its equilibrium expression, for the fifth equality we perform another change in the order of integration. Simplifying yields the definition of expected revenue under the uniform price protocol.

Proposition 1. *Bidding marginal prices is a dominant strategy for investors.*

Proof: Let $\mathcal{P}$ denote the set of marginal prices chosen by the government. A marginal price $P_c(\theta; p)$, depends on the realization of θ, for a given aggregate bid schedule. As θ is drawn from a discrete distribution with finite support, the set $\mathcal{P}$ is itself finite.

Let the shock θ take two possible values, further, let the associated marginal prices be $P_{c,1} > P_{c,2}$. Consider a bid with price p. Suppose for the sake of contradiction that a bid function (p_1, p_2) that has $p_1 = P_{c,1} > p_2 > P_{c,2}$ dominates (p_1, p_2') that has $p_1 = P_{c,1} > p_2' = P_{c,2}$.

The first bid is the same across bid functions so we can focus on the second bid. First note that the bid with price p_2 is accepted with the same probability of a bid with price $p_2' = P_{c,2}$, as there is no marginal price chosen in between $P_{c,1}$ and $P_{c,2}$. It follows that

$$Pr[P_c(\theta) \leq p_2] = Pr[P_c(\theta) \leq p_2'] = Pr[P_c(\theta) \leq P_{c,2}]$$

Second, the value of debt only depends on the marginal price, that is not affected by an individual infinitesimal dealer. In particular we have $Q(\ell(P_{c,1})) \geq Q(\ell(P_{c,2}))$. That is, bidding p_2 instead of p_2' does not affect the value of debt, the investors payoff if the bid is executed. The cost associated with bidding $p_2 > P_{c,2}$, however, is greater than bidding $p_2' = P_{c,2}$. The dealer's unitary profit for this bid is then:

$$Pr[P_c(\theta) \leq P_{c,2}]\Big(Q(\ell(P_{c,2})) - \phi(p_2, P_c(\theta)|\ell(P_{c,2}))\Big)$$

For a discriminatory price protocol $\phi(p_2, \cdot) = p_2$ and

$$Pr[P_c(\theta) \leq P_{c,2}]\Big(Q(\ell(P_{c,2})) - p_2\Big) < Pr[P_c(\theta) \leq P_{c,2}]\Big(Q(\ell(P_{c,2})) - p_2'\Big) \tag{1}$$

Let the uniform price protocol be the limiting case when $\epsilon \to 0$ as follows: $\epsilon p_2 + (1 -$

$\epsilon)P_{c,2}$, then in the limit,

$$\epsilon p_2 + (1-\epsilon)P_{c,2} - (\epsilon p_2' + (1-\epsilon)P_{c,2}) \to 0 \tag{2}$$

and the government is indifferent between bidding p_2 and p_2'.

Note that (1) and (2) contradict $p = (p_1, p_2)$ dominating $p' = (p_1, p_2')$. It follows that bidding marginal prices is a strictly dominant strategy under a discriminatory price protocol and a weakly dominant strategy under a uniform price protocol. □

Appendix C: Investor Heterogeneity

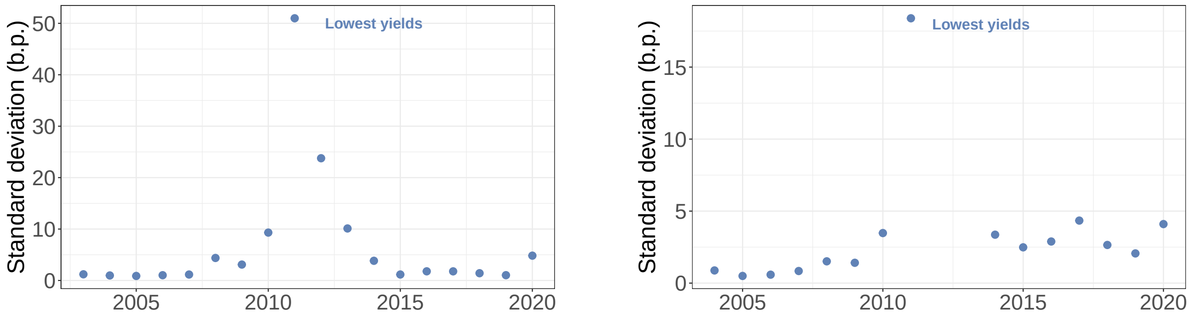
To assess differences across dealers we first look at the variation in the price of the first bid, with the lowest yield (highest price). The first bid in a dealer's bid function is the most likely to be executed as it has the lowest yield. As such, we argue that this bid is also the most informative regarding dealer's characteristics.

Figure 1 presents the standard deviation of the lowest bid yield across dealers, $\{1, \ldots, N\}$, at a given auction, for treasury bills and treasury bonds. More precisely, each point represents the average of such standard deviations across auctions, $\{1, \ldots, M_t\}$, for a given year, t as follows:

$$\overline{SD}_t = \frac{1}{M_t} \sum_{j=1}^{M_t} \sqrt{\sum_{i=1}^{N} \frac{(p_{i1j} - \overline{p}_{1j})^2}{N}}$$

We separate this analysis for bills and bonds as the set of dealers participating in bills and bond auctions are potentially different. Moreover, for bonds, we see a change in protocol during the crisis as well as a hiatus on issuances.

For both securities we see that prior to the crisis the standard deviation is very small. We then see a temporary increase during the crisis period followed by a return to zero afterwards. This pattern is more clear for treasury bills given the continued issuance of these securities during the crisis. Apart from that, the main difference with respect to treasury bonds is that after the crisis the variation does not quite go back to zero, instead it remains at slightly higher levels than before the crisis. This change in pattern occurs at



(a) Treasury Bills (b) Treasury Bonds

Figure 1: Variation between dealers' lowest bid yields

the same time as the protocol for treasury bond auctions switched from discriminatory to uniform price.

Figure 2 presents the standard deviation of bid yields, within a bid function, for each dealer. More precisely, each point represents the average of such standard deviations across auctions, $\{1, \ldots, M_t\}$, for a given dealer, i, and a given year, t, as follows:

$$\overline{SD}_{i,t} = \frac{1}{M_t} \sum_{j=1}^{M_t} \sqrt{\sum_{k=1}^{K_j} \frac{(p_{ikj} - \overline{p}_{ij})^2}{N}}$$

The time series of the average standard deviation as a similar trend across all dealers (across panels): 1) increasing towards 2008; 2) a drop in 2009 before the crisis; 3) higher from 2010 to 2012; 4) a decrease starting in 2013 particularly accentuated in 2014; and, 5) almost flat bid functions from 2014 onward. Note, however, that some dealers have more disperse bid functions than others during the crisis period.

Having said this, the standard deviation across and within investors is at roughly the same magnitude. This leads us to conclude that the pattern we see in Figure 3 of the main text is due to all investors bidding at a wider range of prices, as well as some of them bidding at higher prices and others at lower prices. That is, steeper aggregate bid functions are due, in part, to steeper individual bid functions.

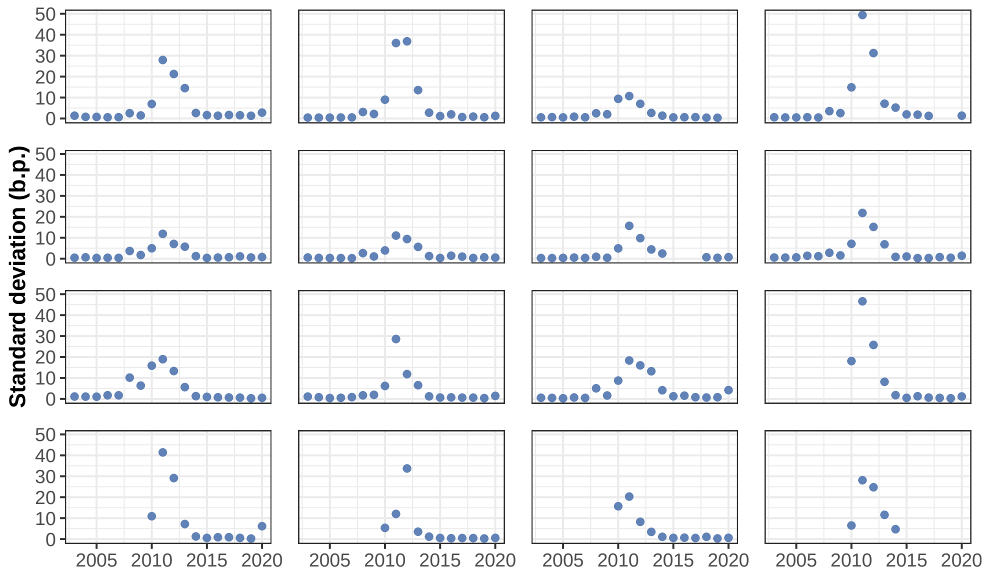


Figure 2: Average standard deviation of bids between dealers

No persistent heterogeneity. Finally, we evaluate whether there are persistent differences across investors. To do so, we rank the first bids, with the lowest yields, across dealers in a given auction. As before, we use the first bid as it is the most informative about differences across dealers. We postulate that if there were persistent differences across investors, we would see a persistent pattern in this ranking. For instance, well informed dealers would likely bid closer to the marginal price of the auction and consistently be ranked lower. Figure 3 depicts the relative ranking over time for each dealer across treasury bill auctions. We focus on treasury bills to highlight this fact due to the continued issuance of this type of securities during the crisis[1]. One can see that a persistent pattern does not seem to exist, in fact, ranking over time seems to be independent of dealer.

The horizontal bars in Figure 3 represent the dealer fixed effect, α_i, in the following regression:

$$R_{it} = \alpha_i + \epsilon_{it}$$

where the dependent variable R_{it} depicts dealers' i ranking in the auction ran at time t. Year fixed effects were not included due to lack of significance. The dealer fixed effects,

[1]The same pattern emerges for treasury bonds.

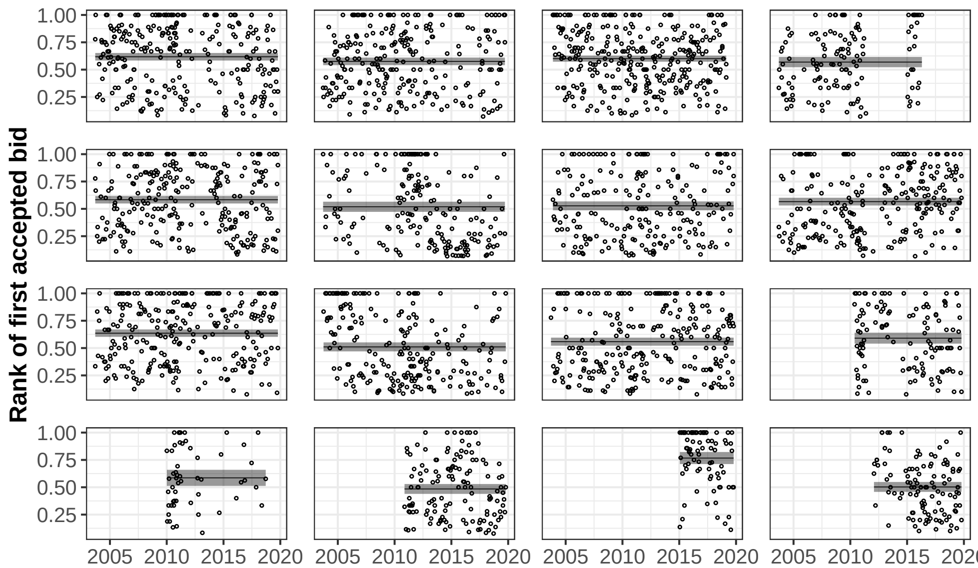


Figure 3: Rank of first bid (if accepted) over time

as seen on the figure are fairly close to each other, with few exceptions, mostly on the lower panels. Particularly, those exceptions tend to be more significant for dealers that participate in auctions during shorter periods of time[2]. Overall, individual and time fixed effects account for less than 5% of the variation of rankings across investors and over time[3].

Appendix D: Two Period Environment, Alternative Specification

Consider the environment described in section 3. Instead of unexpected spending as a random variable, consider a preference shock in the first period. In particular, preferences over streams of consumption are as follows:

$$\mathbb{E}\left[\theta u(c_0) + \beta u(c_1)\right]$$

The taste shock, θ, is privately observed by the government. It is drawn from a continuous distribution with support on $[\theta_L, \theta_H]$ with $\theta_L < \theta_H$ and cdf G. We further assume that

[2]From all participating investors, 6 were not included in the plot as they participated for even shorter periods of time, making it harder to highlight trends in their ranking.

[3]We further test for linear trends within investors across time and verify that they are either not significant or explain less than 10% of the variation of the ranking over time.

$g(\theta) = G'(\theta) > 0$ on $[\theta_L, \theta_H]$.

We use the same parameterization as before with one difference. Suppose that $v^d = y(1 - \exp(-z))$ with z distributed exponentially with cdf $F(z) = 1 - exp(-\mu z)$ and $z = -ln(1 - v^d/y)$. Then $F(v^d) = 1 - exp(\mu ln(1 - v^d/y)) = 1 - (1 - v^d/y)^\mu$ and $F'(v^d) = (\mu/y)(1 - v^d/y)^{\mu-1}$. For $\mu = 1$ this collapses into an uniform distribution on the interval $[0, y]$.

Commitment to a Borrowing Rule

We first tie the hands of the government. Suppose the government could commit to a borrowing rule, θ is observed ex-post and the government commits to it. In particular, the government commits to the optimal borrowing rule under UP, regardless of the protocol being used. That is, $b(\theta) = b(\theta)_{UP}$. By fixing the distribution of b' across protocols, utility in the second period is independent of the protocol. Furthermore, we recover revenue equivalence. With linear utility, welfare is pinned down by

$$\mathbb{E}[\theta\Delta(\theta)] = \mathbb{E}[\theta]\mathbb{E}[\Delta(\theta)] + \mathbb{C}(\theta, \Delta(\theta))$$

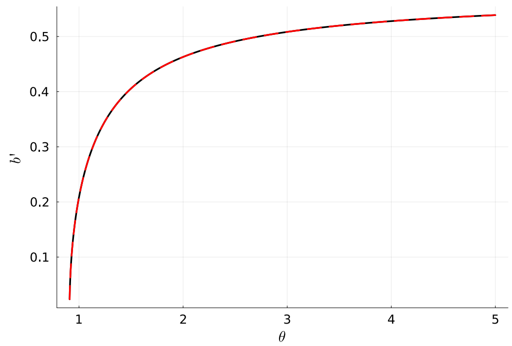
In particular, the difference in welfare across protocols is determined by the covariance term. This term is the insurance component that stems from the curvature introduced by the multiplicative taste shock, θ.

In Figure 4 below we can see how the protocols compare. Panel (a) illustrates the commitment to a borrowing rule as a function of θ. Panel (b) shows us that static dilution is still present, with the bid schedule under a DP lower than the one under a UP. Panel (c) highlights the potential benefits of insurance, higher revenue in bad states at the expense of relative lower revenue in good states. Panel (d) shows us that welfare tends to be higher under DP, particularly when there are large financing needs in the first period.

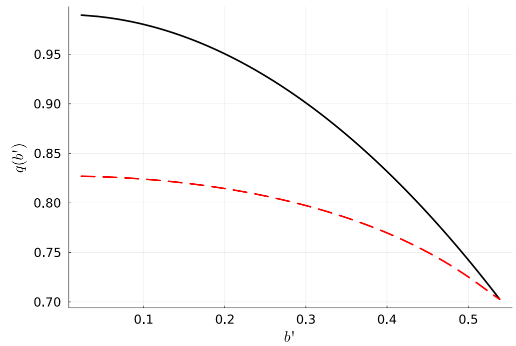
Ex-ante welfare is higher under DP:

$$\mathbb{E}[V(\theta)_{UP}] = 1.754 \quad \mathbb{E}[V(\theta)_{DP}] = 1.761$$

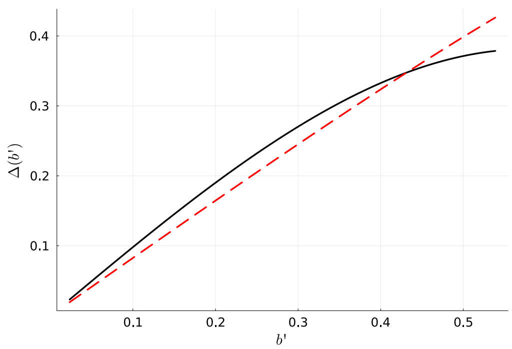
It follows that the covariance term is larger under DP.



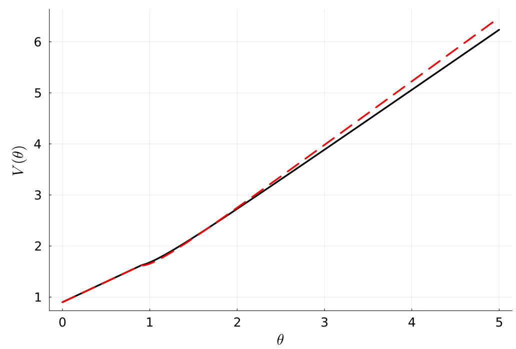
(a) Borrowing Decisions



(b) Bid Schedules



(c) Revenue



(d) Value functions

Figure 4: Comparing outcomes under commitment to $b(\theta)_{UP}$

To evaluate the cost of not committing to the borrowing rule, we let the government choose optimally for each realization of θ, given the price schedule. Note that welfare under a UP will be unchanged as the government was already choosing optimally. As such, the difference in welfare under the DP measures the static dilution that arises from the lack of commitment. This specification of the environment allows us to get a closed form solution detailed below.

Closed Form Solution

The equilibrium under a uniform price protocol is fairly standard to solve. Under the specified functional forms, we can also find a closed-form solution for the fixed point problem described above, between investors' and government's strategies, under a discriminatory price protocol.

Consider linear preferences, such that $u(x) = x$. Let us first characterize the equilibrium under a discriminatory price protocol. An equilibrium requires an actuarially fair price for investors and attaining the maximum in problem (1) of the main text, respectively:

$$p(b) = \frac{1}{1 - G(\theta(b))} \int_{\theta(b)}^{\theta_H} Q(b(\theta)) dG(\theta)$$
$$\theta p(b(\theta)) = \beta F(y - b(\theta))$$

These two conditions together give us a single optimality condition:

$$\frac{\theta}{1 - G(\theta)} \int_{\theta}^{\theta_H} F(y - b(x)) dG(x) = \beta R F(y - b(\theta))$$

Before we solve the equation above, let us just point out that the solution relies on the fact that for a small enough θ the government does not borrow, $b(\theta) = 0$, and so the probability of repayment equals one. As $\theta \to 0$ the benefit of borrowing goes to zero as U only depends on consumption in the second period. Below we show that borrowing is non-decreasing in θ.

Proposition 1 (Monotonicity of b)**.** *If u is strictly increasing and concave, $\beta \in (0, 1)$, and $f(v^d) = F'(v^d) > 0$ on $[\underline{v}, \overline{v}]$, then $b(\theta; p)$ is non-decreasing in θ.*

Proof: Suppose, for the sake of contradiction that $b(\theta)$ is strictly decreasing in θ.

A government chooses b such that:

$$\theta u'(y + \Delta(b) - b_0)\Delta'(b) = \beta F(u(y - b))u'(y - b) \tag{3}$$

Let $\theta_2 > \theta_1 > 0$ and $b(\theta_1)$ and $b(\theta_2)$ be the optimal choices associated with θ_1 and θ_2, respectively. Note that the marginal benefit of borrowing (left-hand side) is non-increasing in b. First, $u'(y + \Delta(b) - b_0)$ is non-increasing in $\Delta(b)$ as u is concave by assumption;

$\Delta(b)$ is a concave function of b as the price schedule is a non-increasing function of b and so $\Delta'(b)$ is also non-increasing. Further, optimality requires that $\Delta'(b) \geq 0$ along the equilibrium path. As $\theta_2 > \theta_1$ and $b(\theta_1) > b(\theta_2)$, it follows that:

$$\theta_2 u'(y + \Delta(b(\theta_2)) - b_0)\Delta'(b(\theta_2)) > \theta_1 u'(y + \Delta(b(\theta_1)) - b_0)\Delta'(b(\theta_1)) \tag{4}$$

Pick a value of y (or $\overline{v}$) such that $F(u(y - b(\theta_2))) = 1$, that is, $u(y - b(\theta_2)) > u(y - b(\theta_1)) \geq \overline{v}$. Then, for $\theta \in \{\theta_1, \theta_2\}$ the government never defaults and $Q(b(\cdot)) = R^{-1}$.

The marginal cost of borrowing, when default is a zero probability event, is non-decreasing in b, as u is concave and $F(\cdot)$ is constant and equal to 1. As $b(\theta_1) > b(\theta_2)$, it follows that:

$$\beta F(u(y - b(\theta_2)))u'(y - b(\theta_2)) \leq \beta F(u(y - b(\theta_1)))u'(y - b(\theta_1))$$

Note that equation (3) then requires:

$$\theta_2 u'(y + \Delta(b(\theta_2)) - b_0)\Delta'(b(\theta_2)) \leq \theta_1 u'(y + \Delta(b(\theta_1)) - b_0)\Delta'(b(\theta_1))$$

which contradicts equation (4). $\square$

Under a discriminatory price protocol, we have established that, with linear preferences, for any θ that has the government borrowing in equilibrium, it must be that:

$$\theta \frac{R^{-1}}{1 - G(\theta)} \int_{\theta}^{\theta_H} F(\underline{v}^d(b(x))dG(x) = \beta F(y - b(\theta))$$

Let $n(\theta) \equiv F(y - b(\theta))$, denote the probability of repayment at θ. Then, the equation above is

$$\theta \int_{\theta}^{\theta_H} n(x) \frac{dG(x)}{1 - G(\theta)} = \beta R \, n(\theta)$$

Set $N(\theta) \equiv \int_{\theta}^{\theta_H} n(x)dG(x)$ and $N'(\theta) = -n(\theta)dG(\theta)$. Then,

$$\theta N(\theta) = -\beta R \frac{1 - G(\theta)}{dG(\theta)} N'(\theta) \iff \frac{N'(\theta)}{N(\theta)} = -\frac{\theta}{\beta R} \frac{dG(\theta)}{1 - G(\theta)}$$

For an exponentially distributed θ, we have $G(x) = 1 - exp(-\lambda x)$ and $dG(x) = \lambda exp(-\lambda x)$.

$$\frac{N'(\theta)}{N(\theta)} = -\frac{\theta}{\beta R}\frac{1-G(\theta)}{dG(\theta)} \iff \log(N(\theta)) = -\frac{\theta^2\lambda}{2\beta R} + C \implies N(\theta) = K.exp\left(-\frac{\theta^2\lambda}{2\beta R}\right)$$

where $K = exp(C)$. Taking derivatives we get:

$$N'(\theta) = -K\frac{\theta}{\beta R}exp\left(-\frac{\theta^2\lambda}{2\beta R}\right)$$

Recalling that $N'(\theta) = -n(\theta)dG(\theta)$, by definition this is equivalent to:

$$n(\theta) = K\frac{\theta}{\beta R}exp\left(\lambda\theta - \frac{\theta^2\lambda}{2\beta R}\right)$$

This must be true for some $K > 0$. To determine the value of K, we make some assumptions about the nature of the equilibrium. In general, an equilibrium of the kind we posit always exists. We look for equilibria in which 1) there is a $\hat{\theta}$, such that the government's first order condition holds at $b' = 0$ (and therefore $n(\hat{\theta}) = 1$), and 2) at this $\hat{\theta}$, it is the case that $n'(\hat{\theta}) = 0$. The second condition selects a specific θ. In particular, it selects the lowest possible one. We begin solving for $\hat{\theta}$ and K by examining the implications of $n'(\hat{\theta}) = 0$. The derivative of $n(\theta)$ is:

$$n'(\theta) = K\frac{1}{\beta R}exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right) + K\frac{\theta}{\beta R}\left(\lambda - \frac{\lambda\theta}{\beta R}\right)exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right)$$

Collect terms to rewrite this as:

$$n'(\theta) = K\frac{1}{\beta R}\left(1 + \lambda\theta\left(1 - \frac{\theta}{\beta R}\right)\right)exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right)$$

Since the collection of terms outside the big parentheses are all positive, we see that this is a parabola that opens down. Setting it equal to 0 yields:

$$1 + \lambda\theta - \frac{\lambda\theta^2}{\beta R} = 0$$

We will want the right root of this (so that $n'(\theta)$ is appropriately negative for $\theta \geq \hat{\theta}$). The above can be rewritten as:

$$\theta^2 - \beta R\theta - \frac{\beta R}{\lambda} = 0$$

Then $\hat{\theta}$ is given by:

$$\hat{\theta} = \frac{\beta R + \sqrt{(\beta R)^2 + 4\frac{\beta R}{\lambda}}}{2}$$

Finally, having solved for $\hat{\theta}$ in terms of parameters, we can quickly solve for K as the solution to:

$$1 = n(\hat{\theta}) = K\frac{\hat{\theta}}{\beta R}exp\left(\lambda\hat{\theta} - \frac{\lambda\hat{\theta}^2}{2\beta R}\right)$$

So:

$$K = \frac{\beta R}{\hat{\theta}}exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right)$$

Then, $n(\theta)$ becomes:

$$n(\theta) = \frac{\beta R}{\hat{\theta}}exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right)\frac{\theta}{\beta R}exp\left(\lambda\theta - \frac{\theta^2\lambda}{2\beta R}\right)$$

which can be simplified to:

$$\begin{aligned} n(\theta) &= \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right) \\ &= \frac{\theta}{\hat{\theta}}\, exp\left(-\frac{\lambda}{\beta R}(\theta - \hat{\theta})\left(\frac{\theta + \hat{\theta}}{2} - \beta R\right)\right) \end{aligned}$$

Given a functional form of $F(\cdot)$, this can then be mapped back to choices of b' using the definition:

$$n(\theta) = F(y - b'(\theta))$$

Suppose that $v^d = y(1 - \exp(-z))$ where z is distributed exponentially with cdf $F(z) = 1 - exp(-\mu z)$ and $z = -ln(1 - v^d/y)$. Then $F(v^d) = 1 - exp(\mu ln(1 - v^d/y)) = 1 - (1 - v^d/y)^\mu$ and $F'(v^d) = (\mu/y)(1 - v^d/y)^{\mu-1}$. When $v^d = y - b(\theta)$, the term $1 - v^d/y$

becomes:

$$1 - \frac{y - b(\theta)}{y} = \frac{y - y + b(\theta)}{y} = \frac{b(\theta)}{y}$$

Using the optimality condition we get:

$$\begin{aligned}
&\beta F(y - b(\theta)) = \frac{\theta R^{-1}}{exp(-\lambda\theta)} N(\theta) \iff \\
&\iff \beta \left(1 - \left(\frac{b(\theta)}{y}\right)^{\mu}\right) = \frac{\theta R^{-1}}{exp(-\lambda\theta)} \frac{\beta R}{\hat{\theta}} exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right) exp\left(-\frac{\theta^2\lambda}{2\beta R}\right) \\
&\iff 1 - \left(\frac{b(\theta)}{y}\right)^{\mu} = \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right) \\
&\iff b(\theta) = y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}
\end{aligned}$$

Recall that $p(b(\theta)) = \frac{\beta}{\theta} F(y - b(\theta))$, it then follows that:

$$\begin{aligned}
p(b(\theta)) &= \frac{\beta}{\theta}\left(1 - \left(\frac{b(\theta)}{y}\right)^{\mu}\right) \\
&= \frac{\beta}{\theta}\left(1 - \left[\frac{y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}}{y}\right]^{\mu}\right) \\
&= \frac{\beta}{\theta}\left(\frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right) \\
&= \frac{\beta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right) \\
&= p(\theta)
\end{aligned}$$

Under a uniform price protocol, an equilibrium with positive borrowing requires:

$$\begin{aligned}
&\theta\Delta'(b(\theta)) = \beta F(y-b(\theta)) \iff \\
\iff &\theta\left[Q(b(\theta)) + \frac{\partial Q(b(\theta))}{db(\theta)}b(\theta)\right] = \beta F(y-b(\theta)) \\
\iff &\theta\left[R^{-1}F(y-b(\theta)) + R^{-1}F'(y-b(\theta))b(\theta)\right] = \beta F(y-b(\theta)) \\
\iff &\theta\left[1 - \frac{F'(y-b(\theta))}{F(y-b(\theta))}b(\theta)\right] = \beta R \\
\iff &\theta\frac{F'(y-b(\theta))}{F(y-b(\theta))}b(\theta) = \theta - \beta R \\
\iff &\theta\frac{\frac{\mu}{y}\left(\frac{b(\theta)}{y}\right)^{\mu-1}}{1-\left(\frac{b(\theta)}{y}\right)^{\mu}}b(\theta) = \theta - \beta R \\
\iff &\theta\frac{\mu}{y^{\mu}}b(\theta)^{\mu} = (\theta-\beta R) - (\theta-\beta R)b(\theta)^{\mu}\frac{1}{y^{\mu}} \\
\iff &b(\theta)^{\mu}\left(\theta\frac{\mu}{y^{\mu}} + \theta\frac{1}{y^{\mu}} - \beta R\frac{1}{y^{\mu}}\right) = \theta - \beta R \\
\iff &b(\theta) = \left(\frac{\theta-\beta R}{\theta(1+\mu)-\beta R}\right)^{\frac{1}{\mu}} y
\end{aligned}$$

We have established that under a uniform price protocol investors only bid marginal prices, hence:

$$\begin{aligned}
p(b(\theta)) &= R^{-1}F(y-b(\theta)) \\
&= R^{-1}\left(1-\left(\frac{\theta-\beta R}{\theta(1+\mu)-\beta R}\right)\right) \\
&= R^{-1}\left(\frac{\mu\theta}{\theta(1+\mu)-\beta R}\right) \\
&= p(\theta)
\end{aligned}$$

Summing up, for a **uniform price auction**:

$$b(\theta) = \left(\frac{\theta-\beta R}{\theta(1+\mu)-\beta R}\right)^{\frac{1}{\mu}} y$$

$$p(\theta) = R^{-1}\left(\frac{\mu\theta}{\theta(1+\mu) - \beta R}\right)$$

And for a **discriminatory price auction**:

$$b(\theta) = y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}$$

$$p(\theta) = \frac{\beta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)$$

Appendix E: Robustness

Utility

Let us first see what happens under different utility functions. All other parameters are the same as before.

$$\begin{aligned}
\text{Linear Utility}: \ & \mathbb{E}[V(\theta)_{UP}] = 1.754 > \mathbb{E}[V(\theta)_{DP}] = 1.698 \\
\text{Log Utility}: \ & \mathbb{E}[V(\theta)_{UP}] = -0.0986 < \mathbb{E}[V(\theta)_{DP}] = -0.0973 \\
\text{CRRA}, \gamma = 2: \ & \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
\text{CRRA}, \gamma = 4: \ & \mathbb{E}[V(\theta)_{UP}] = -0.8065 < \mathbb{E}[V(\theta)_{DP}] = -0.8059 \\
\text{CRRA}, \gamma = 8: \ & \mathbb{E}[V(\theta)_{UP}] = -0.5146 < \mathbb{E}[V(\theta)_{DP}] = -0.5137
\end{aligned}$$

Distribution of θ

Let us keep CRRA with $\gamma = 2$ and v^d uniformly distributed on $[\underline{v}, \overline{v}]$.

$$\begin{aligned}
\theta \sim Exp(1): \ & \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
\theta \sim U(0,5): \ & \mathbb{E}[V(\theta)_{UP}] = -3.5773 < \mathbb{E}[V(\theta)_{DP}] = -3.5762 \\
\theta \sim U(0,10): \ & \mathbb{E}[V(\theta)_{UP}] = -5.6877 < \mathbb{E}[V(\theta)_{DP}] = -5.6851 \\
\theta \sim N(3,2): \ & \mathbb{E}[V(\theta)_{UP}] = -4.1615 < \mathbb{E}[V(\theta)_{DP}] = -4.1599
\end{aligned}$$

Distribution of v^d

Let us keep CRRA with $\gamma = 2$ and θ exponentially distributed with $\lambda = 1$.

$$v^d \sim U(\underline{v}, \overline{v}) : \ \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259$$
$$v^d \sim N(u(0.2) = -5, 1.5) : \ \mathbb{E}[V(\theta)_{UP}] = -2.0254 < \mathbb{E}[V(\theta)_{DP}] = -2.0248$$

Output Growth

Let us keep CRRA with $\gamma = 2$, θ exponentially distributed with $\lambda = 1$ and v^d uniformly distributed.

$$y_1 = y_0 : \ \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259$$
$$y_1 = 1.05 \times y_0 : \ \mathbb{E}[V(\theta)_{UP}] = -1.9677 < \mathbb{E}[V(\theta)_{DP}] = -1.9671$$
$$y_1 = 0.95 \times y_0 : \ \mathbb{E}[V(\theta)_{UP}] = -2.0896 < \mathbb{E}[V(\theta)_{DP}] = -2.0891$$

Budget Deficits

Instead of considering a multiplicative taste shock we now look at what would happen if instead uncertainty is regarding a budget deficit, θ as follows:

$$c = y + \Delta(b(\theta)) - b_0 - \theta$$

Let us keep CRRA with $\gamma = 2$ and v^d uniformly distributed. θ is exponentially distributed with $\lambda = 1$ and truncated to the interval $[0, 1]$.

$$\mathbb{E}[V(\theta)_{UP}] = -2.8976 < \mathbb{E}[V(\theta)_{DP}] = -2.8952$$

Appendix F: Computational Details

Environment: Additional Elements

The private exogenous state includes a vector $\mathbf{m}$ of preference shocks for the government that is i.i.d. over time. These preference shocks enter additively in the government's decision problems. They are unbounded and therefore ensure that every feasible action is played with positive probability in equilibrium. Introducing these shocks is like introducing randomization, ensuring convergence – that an equilibrium exists [4]. These shocks are otherwise small. The preference shocks **m** are distributed according to a Generalized Type One Extreme Value distribution with scale parameter σ_m and correlation parameter ρ_m. These distributions are chosen for their computational tractability[5].

When the government issues debt, it incurs an issuance cost $i(s, B, B') \geq 0$. This is a standard feature in models with long term debt and positive recovery rates (see Dvorkin et al. (2021) or Chatterjee and Eyigungor (2015)). Without these adjustment costs, the government has an incentive to issue very large amounts of debt when default is imminent in order to extract the value of existing bondholders' securities. This type of "maximum" dilution behavior is counterfactual. As such, issuance costs are added to the model to prevent it from occurring in equilibrium. Quantitatively, the amount spent financing the issuance costs ends up being small. The issuance cost function is as in Fourakis (2023)[6]. This function imposes a strict limit on the one period ahead default probability from which issuing costs are positive and is continuous in the scale of the issuance. The purpose of these issuance costs is to prevent a behavior Chatterjee and Eyigungor (2015) termed "maximum dilution."

Solving the Model

The set of objects used to solve the model numerically and assess convergence are as follows:

[4]These preference shocks have the same role as the m shock introduced in Chatterjee and Eyigungor (2012).

[5]Specifically, both choice probabilities and ex ante expected values can be written analytically in terms of the values associated with the choices. We set ρ_m following Dvorkin et al. (2021). We then set the scale parameter at a small number that still ensures convergence, half of that of Dvorkin et al. (2021).

[6]A sine wave shifted and scaled to rise from 0 to 1 as it travels from the threshold, p_d, to 1:

$$i\left(s, B, B'\right) = \begin{cases} 0 & B' \leq \hat{B} \text{ or } \Pr\left(d'^{\star} = 1\right) \leq p_d \\ \frac{1}{2}\left(1 + \sin\left(\pi\left(\frac{\Pr(d'^{\star}=1)-p_d}{1-p_d} - \frac{1}{2}\right)\right)\right) & B' > \hat{B} \text{ and } \Pr\left(d'^{\star} = 1\right) > p_d \end{cases}$$

where $\hat{B} = \max\{(1-\lambda)B, 0\}$.

1. The continuation value functions $W(s, T, B, B')$ and $W^D(s, T, B)$, given by:

$$W(s, T, B, B') = \mathbb{E}[V(s', T', \mathbf{m}, B, B')|s]$$
$$W^D(s, T, B) = \mathbb{E}[V^D(s', T', \mathbf{m}, B)|s]$$

2. The price functions $Q(s, B')$ and $Q^D(s, B)$ and the expected probability of default $\delta(s, B')$.

In short, these are the continuation value functions, the price functions, and the expected probability of default. Note that there are other price and value functions (including the bid function in the discriminatory price protocol), but they can be derived based on the above set of objects and within-period optimization. We use the above set as the list to assess convergence.

These objects are defined on grids of their arguments. In particular, we have the following sets that we will need to define grids for:

1. $s \in \mathcal{S}$, that defines GDP and expected public spending.
   (a) For the grid of GDP values, $y(s)$, we use 23 points evenly spaced in logs spread across a space spanning six of the logged variable's long run standard deviations and centered at its mean:

   $$[\mathbb{E}[\log(y(s))] - 3\sigma[\log(y(s))], \mathbb{E}[\log(y(s))] + 3\sigma[\log(y(s))]]$$

   (b) For the grid of expected public spending values, $g(s)$, we use 17 points evenly spaced in logs spread across a space spanning six of the logged variable's long run standard deviations and centered at its mean:

   $$[\mathbb{E}[\log(g(s))] - 3\sigma[\log(g(s))], \mathbb{E}[\log(g(s))] + 3\sigma[\log(g(s))]]$$

2. $B \in \mathcal{B}$: for the grid of b we use 241 evenly spaced points on [0, 1.2].

3. $T \in \mathcal{T}$, that defines surprise budget spending: for the grid of $\theta(T)$ we use 31 points evenly spaced spanning six of the logged variable's long run standard deviations and centered at one (the average log is zero).

Given a guess for the set of objects listed above, in order to generate a new guess, the iteration proceeds as follows:

1. Using the baseline set of objects, and given the restructuring structure upon regaining access to financial markets, generate new guesses for $W^D(s, T, B)$ and $Q^D(s, B)$.
2. Using the baseline set of objects, and those defined in the previous step, solve the government's problem when it enters a period in good standing. Use the solution to generate new guesses of $W(s, T, B, B')$, $Q(s, B')$ and $\delta(s, B')$.
3. Check the sup-norm distance between all objects. If it is less than 10^{-5}, stop. Otherwise, update guesses using rules of the form

$$f_{next}(\cdot) = \xi_j f_{old}(\cdot) + (1 - \xi_j) f_{new}(\cdot)$$

   where $j \in \{V, Q\}$, and return to step 1.

   This type of rule updates the old guess by moving fraction $(1 - \xi_j)$ of the distance towards the new guess. In general, to ensure convergence, updates of the the price functions tend to require more smoothing than those of the value functions. Moreover, solving the government's problem in good standing under a discriminatory price protocol also requires smoothing for the update of bid schedules and auction revenue.

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