Macroeconomic Theory (Econ 8105) Recitations

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[1]The following are a combination of Professor Tim Kehoe's lectures and Sang Min Lee's recitations from Fall 2018, together with my own notes.

Contents

Introduction to Dynamic General Equilibrium

Economic Environment

Pure Exchange Economy

1. Set of commodities $k = 1, \ldots, l$
2. Set of consumers $i = 1, \ldots, n$
   (a) a consumption set X^i, typically $X^i \subset \mathbb{R}^l_+$
   (b) initial endowment $e^i \in \mathbb{R}^l_+$
   (c) utility function $u^i : X^i \to \mathbb{R}$

Economy: $\mathcal{E} = \{e^i, u^i\}_{i=1}^n$

Arrow-Debreu Equilibrium (ADE)

In an Arrow-Debreu market structure markets are open in period 0. Consumers meet and trade future contracts among themselves (the underlying asset of the future contracts is the consumption good).

Prices p_t are current prices (in period zero) of future consumption in period t, hence discounted prices.

Definition 1 (ADE) *An ADE is a sequence of prices $\hat{p} = \{\hat{p}_t\}_{t=0}^\infty \in \mathbb{R}^\infty_{++}$ and consumption levels $\hat{c}^i = \{\hat{c}^i_t\}_{t=0}^\infty \in \mathbb{R}^\infty_+$, $i = 1, \ldots, n$, such that:*

1. *Given $\hat{p}$, consumer i chooses $\hat{c}^i$ to solve, $\forall i \in 1, \ldots, n$:*

$$\begin{aligned} &\max_{\{c^i_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u^i(c^i_t) \\ s.t. \quad &\sum_{t=0}^\infty \hat{p}_t c^i_t \leqslant \sum_{t=0}^\infty \hat{p}_t e^i_t \\ &c^i_t \geqslant 0 \quad \forall t \end{aligned}$$

2. *Markets clear:*

$$\sum_{i=1}^n \hat{c}^i_t = \sum_{i=1}^n e^i_t, \quad \forall t$$

Characterization

$$\mathcal{L} = \sum_{t=0}^\infty \beta^t u^i(c^i_t) + \lambda^i \left(\sum_{t=0}^\infty \hat{p}_t e^i_t - \sum_{t=0}^\infty \hat{p}_t c^i_t \right) + \sum_{t=0}^\infty \phi^i_t c^i_t$$

(c^i_t) $\quad \beta^t u^i_{\hat{c}}(t) - \lambda^i \hat{p}_t + \phi^i_t = 0$

(λ^i) $\quad \sum_{t=0}^\infty \hat{p}_t e^i_t \geqslant \sum_{t=0}^\infty \hat{p}_t c^i_t \quad \lambda^i \geqslant 0 \quad \lambda^i \left(\sum_{t=0}^\infty \hat{p}_t e^i_t - \sum_{t=0}^\infty \hat{p}_t \hat{c}^i_t \right) = 0$

(ϕ^i_t) $\quad \hat{c}^i_t \geqslant 0 \quad \phi^i_t \geqslant 0 \quad \phi^i_t \hat{c}^i_t = 0$

Assumptions

1. The utility function u^i is differentiable ($\forall i$);
2. The utility function u^i is strictly increasing ($\forall i$);
3. The utility function u^i Satisfies Inada Conditions ($\forall i$):
   - $\lim_{c\to\infty} u_c^i(t) = 0$
   - $\lim_{c\to 0} u_c^i(t) = \infty$

Note that,

A.1 is needed to get the first order conditions.

A.2 gives us that the budget constraint is binding as otherwise the agent could increase consumption and be better off.

A.3 gives us that $c_t^i > 0$ as the marginal utility of increasing c_t^i slightly above zero is infinity.

With the 3 assumptions together the first order conditions give us the Euler equation:

$$\frac{u_{\hat{c}}^i(t)}{\beta u_{\hat{c}}^i(t+1)} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$$

Sequential Markets Equilibrium (SME)

In a Sequential markets structure markets for consumption and for bonds are open every period. Consumers meet and trade among themselves.

Definition 2 (SME) *A SME is a sequence of prices $\hat{r} = \{\hat{r}_t\}_{t=0}^{\infty}$, consumption levels $\hat{c}^i = \{\hat{c}_t^i\}_{t=0}^{\infty}$ and asset holdings $b = \{b_t^i\}_{t=0}^{\infty}$, $i = 1, \ldots, n$, such that:*

1. *Given $\hat{r}$, consumer i chooses $\hat{c}^i$ and $\hat{b}^i$ to solve, $\forall i \in 1, \ldots, n$:*

$$\begin{aligned} &\max_{\{c_t^i, b_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) \\ s.t.\quad & c_0^i + b_1^i \leqslant e_0^i \\ & c_t^i + b_{t+1}^i \leqslant e_t^i + b_t^i(1+\hat{r}_t),\ \forall t \geqslant 1 \\ & c_t^i \geqslant 0 \\ & b_{t+1}^i \geqslant -B, \quad B > 0 \end{aligned}$$

2. *Markets clear:*

$$\sum_{i=1}^{n} \hat{c}_t^i = \sum_{i=1}^{n} e_t^i, \quad \forall t \geqslant 1 \qquad \text{(Goods)}$$

$$\sum_{i=1}^{n} \hat{b}_t^i = 0, \quad \forall t \geqslant 1 \qquad \text{(Bonds)}$$

Characterization (under A.1-A.3)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) + \lambda_0^i(e_0^i - c_0^i - b_0^i) + \sum_{t=0}^{\infty} \lambda_t^i \left(e_t^i + b_t^i(1+\hat{r}_t) - c_t^i - b_{t+1}^i\right)$$

(c_t^i) $\quad \beta^t u_{\hat{c}}^i(t) = \lambda_t^i$

(b_{t+1}^i) $\quad \lambda_{t+1}^i(1+\hat{r}_{t+1}) = \lambda_t^i$

(λ_t^i) $\quad \hat{c}_t^i + \hat{b}_{t+1}^i = e_t^i + \hat{b}_t^i(1+\hat{r}_t)$

The Euler equation is then,

$$\frac{u_{\hat{c}}^i(t)}{\beta u_{\hat{c}}^i(t+1)} = 1 + \hat{r}_{t+1}$$

And the transversality condition is:

$$\lim_{t\to\infty} \lambda_t^i \hat{b}_{t+1}^i = \lim_{t\to\infty} \beta^t u_{\hat{c}}^i(t)\hat{b}_{t+1}^i = 0$$

Equivalence between ADE and SME

Stating and proving the equivalence theorem is left as an exercise for future problem sets and/or exams. Note that when showing equivalence you need to show it both ways, i.e. if you have an ADE how do you get a SME and vice-versa.

Hint: Which direction is this?

$$\hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}} - 1$$

Social Planner's Problem

$$\begin{aligned}
&\max_{\{\{c_t^i\}_{t=0}^{\infty}\}_{i=1}^{n}} \sum_{i=1}^{n} \alpha^i \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) \\
\text{s.t.} \quad & \sum_{i=1}^{n} c_t^i = \sum_{i=1}^{n} e_t^i \quad \forall t \\
& c_t^i \geqslant 0 \quad \forall i, t
\end{aligned}$$

Usually it is also imposed that $\alpha^i \geqslant 0$, $\forall i$ and $\exists \alpha^i > 0$. Otherwise the problem would be trivial. Why?

Where are we headed?

$$(ADE \iff SME) \iff (SPP \iff DP)$$

Pareto Efficiency (PE)

Definition 3 (Feasibility) *An allocation* $\mathcal{C} \equiv \{\{c_t^i\}_{t=0}^{\infty}\}_{i=1}^{n}$ *is Feasible if:*

$$\sum_{i=1}^{n} c_t^i \leqslant \sum_{i=1}^{n} e_t^i \ \forall i \quad \wedge \quad (\textit{in our setting}) \ \ c_t^i \geqslant 0 \ \forall i, t$$

Definition 4 (Pareto Efficiency) *An allocation* $\mathcal{C} \equiv \{\{c_t^i\}_{t=0}^{\infty}\}_{i=1}^{n}$ *is Pareto Efficient if it is feasible and there* ***does not exist*** *another feasible allocation* $\hat{\mathcal{C}} \equiv \{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i=1}^{n}$ *such that:*

$$\sum_{t=0}^{\infty} \beta^t u^i(\hat{c}_t^i) \geqslant \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) \ \forall i \quad \wedge \quad \exists j \ s.t. \ \sum_{t=0}^{\infty} \beta^t u^j(\hat{c}_t^j) > \sum_{t=0}^{\infty} \beta^t u^j(c_t^j)$$

Theorem 1 $\mathcal{C}$ *is Pareto Efficient if and only if* $\exists\{\alpha^i\}_{i=1}^{n}$ *such that* $\mathcal{C}$ *solves the Social Planner's Problem.*

(Assumptions needed for the theorem above. However, standard utility satisfies the assumptions. You will cover this in more detail on the second mini of Microeconomics Theory.)

First Welfare Theorem (FWT)

Theorem 2 (FWT) *Let us assume that* u^i *is strictly increasing for all* i. *If* $\hat{\mathcal{C}}$ *and* $\hat{\mathcal{P}} = \{p_t\}_{t=0}^{\infty}$ *characterize an ADE then* $\hat{\mathcal{C}}$ *is Pareto Efficient.*

Proof of FWT:

Suppose for the sake of contradiction that $\hat{\mathcal{C}}$ is not Pareto Efficient. Then there exists a feasible allocation $\tilde{\mathcal{C}}$ such that:

$$\sum_{t=0}^{\infty} \beta^t u^i(\tilde{c}_t^i) \geqslant \sum_{t=0}^{\infty} \beta^t u^i(\hat{c}_t^i) \ \forall i \quad \wedge \quad \exists j \text{ s.t. } \sum_{t=0}^{\infty} \beta^t u^j(\tilde{c}_t^j) > \sum_{t=0}^{\infty} \beta^t u^j(\hat{c}_t^j)$$

<u>Claim 1</u>: For agent j it follows that $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^j > \sum_{t=0}^{\infty} \hat{p}_t e_t^j$

<u>Proof of Claim 1</u>: Suppose for the sake of contradiction that $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^j \leqslant \sum_{t=0}^{\infty} \hat{p}_t e_t^j$, then agent j is not maximizing his/her utility as $\{\tilde{c}_t^j\}_{t=0}^{\infty}$ is affordable and preferred to $\{\hat{c}_t^j\}_{t=0}^{\infty}$. □

<u>Claim 2</u>: For all i it follows that $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^i \geqslant \sum_{t=0}^{\infty} \hat{p}_t e_t^i$

<u>Proof of Claim 2</u>: Suppose for the sake of contradiction that $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^i < \sum_{t=0}^{\infty} \hat{p}_t e_t^i$, then construct $\{\bar{c}_t^i\}_{t=0}^{\infty}$ as follows:

$$\bar{c}_0^i \equiv \tilde{c}_0^i + \frac{\sum_{t=0}^{\infty} \hat{p}_t e_t^i - \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^i}{\hat{p}_0}$$
$$\bar{c}_t^i \equiv \tilde{c}_t^i$$

Note that:

$$\sum_{t=0}^{\infty} \beta^t u^i(\bar{c}_t^i) > \sum_{t=0}^{\infty} \beta^t u^i(\tilde{c}_t^i) \geqslant \sum_{t=0}^{\infty} \beta^t u^i(\hat{c}_t^i)$$

Hence, each agent i is not maximizing his/her utility as $\{\bar{c}_t^i\}_{t=0}^{\infty}$ is affordable and preferred to $\{\hat{c}_t^i\}_{t=0}^{\infty}$. □

By Claim 1 and 2, adding up the inequalities across agents it follows that:

$$\sum_{i=1}^{n}\sum_{t=0}^{\infty}\hat{p}_t\tilde{c}_t^i > \sum_{i=1}^{n}\sum_{t=0}^{\infty}\hat{p}_t e_t^i$$
$$\sum_{t=0}^{\infty}\hat{p}_t\sum_{i=1}^{n}\tilde{c}_t^i > \sum_{t=0}^{\infty}\hat{p}_t\sum_{i=1}^{n}e_t^i$$

Note that if we multiply the resources constraint in each period by its price and add them up across t we get:

$$\sum_{t=0}^{\infty}\hat{p}_t\sum_{i=1}^{n}\tilde{c}_t^i \leqslant \sum_{t=0}^{\infty}\hat{p}_t\sum_{i=1}^{n}e_t^i$$

Which is a contradiction. Hence, $\hat{\mathcal{C}}$ is Pareto Efficient. □

Second Welfare Theorem (SWT)

Theorem 3 (SWT) *Let us assume that u^i is strictly increasing, strictly concave and satisfies Inada conditions for all i. If $\hat{\mathcal{C}}$ is Pareto Efficient then there exists $\hat{\mathcal{P}} = \{\hat{p}_t\}_{t=0}^{\infty}$ and $\hat{\mathcal{T}} = \{\hat{\tau}_i\}_{i=1}^{n}$ such that $(\mathcal{C}, \mathcal{P}, \mathcal{T})$ characterize an ADE with transfers.*

Definition 5 (ADE with transfers) *An ADE with transfers is a sequence of prices $\hat{p} = \{\hat{p}_t\}_{t=0}^{\infty} \in \mathbb{R}_{++}^{\infty}$, consumption levels $\hat{c}^i = \{\hat{c}_t^i\}_{t=0}^{\infty} \in \mathbb{R}_{+}^{\infty}$, $i = 1, \ldots, n$, and transfers $\hat{T} = \{\hat{\tau}_i\}_{i=1}^{n}$, such that:*

1. *Given $\hat{p}$ and $\hat{T}$, consumer i chooses $\hat{c}^i$ to solve, $\forall i \in 1, \ldots, n$:*

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t u^i(c_t^i)$$
$$s.t. \quad \sum_{t=0}^{\infty}\hat{p}_t c_t^i \leqslant \sum_{t=0}^{\infty}\hat{p}_t e_t^i + \hat{\tau}_i$$
$$c_t^i \geqslant 0 \quad \forall t$$

2. *Markets clear:*

$$\sum_{i=1}^{n}\hat{c}_t^i = \sum_{i=1}^{n}e_t^i, \quad \forall t$$

Proof of SWT: By theorem, the solution of a Social Planner's Problem is Pareto Efficient.

The SPP can be solved as follows:

$$\mathcal{L} = \sum_{i=1}^{n}\alpha^i\sum_{t=0}^{\infty}\beta^t u^i(c_t^i) + \lambda_t\left(\sum_{i=1}^{n}e_t^i - \sum_{i=1}^{n}c_t^i\right)$$

(c_t^i) $\quad \alpha^i\beta^t u_{\hat{c}}^i(t) = \lambda_t$

(λ_t) $\quad \sum_{i=1}^{n}\hat{c}_t^i = \sum_{i=1}^{n}e_t^i$

The ADE with transfers can be solved as follows:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) + \phi^i \left(\sum_{t=0}^{\infty} \hat{p}_t e_t^i + \hat{\tau}^i - \sum_{t=0}^{\infty} \hat{p}_t c_t^i \right)$$

(c_t^i) $\quad \beta^t u_{\hat{c}}^i(t) = \phi^i \hat{p}_t$

(ϕ^i) $\quad \sum_{t=0}^{\infty} \hat{p}_t e_t^i + \hat{\tau}^i = \sum_{t=0}^{\infty} \hat{p}_t c_t^i$

Let $\hat{p}_t = \lambda_t$, $\phi^i = \frac{1}{\alpha^i}$ and $\hat{\tau}^i = \sum_{t=0}^{\infty} \hat{p}_t c_t^i - \sum_{t=0}^{\infty} \hat{p}_t e_t^i$ and note that the first order condition is equal in both problems. □

Overlapping Generation Models

Arrow-Debreu Equilibrium

Definition 6 (ADE) *An ADE is a sequence of prices* $\hat{p} = \{\hat{p}_t\}_{t=1}^{\infty}$ *and consumption levels* $\hat{c} = \{\hat{c}_1^0, \{\hat{c}_t^t, \hat{c}_{t+1}^t\}_{t=1}^{\infty}\}$*, such that:*

1. *Given* $\hat{p}_1$ *and* m*, generation* 0 *chooses* $\hat{c}_1^0$ *to solve:*

$$\begin{aligned} &\max_{c_1^0} \; u_0(c_1^0) \\ s.t. \quad &\hat{p}_1 c_1^0 \leqslant \hat{p}_1 e_1^0 + m \\ &c_1^0 \geqslant 0 \end{aligned}$$

2. *Given* $\hat{p}_t$ *and* $\hat{p}_{t+1}$*, generation* t *chooses* $\hat{c}_t^t$ *and* $\hat{c}_{t+1}^t$ *to solve, for all* $t \in \{1, \ldots, \infty\}$*:*

$$\begin{aligned} &\max_{\{c_t^t, c_{t+1}^t\}} \; u_t(c_t^t, c_{t+1}^t) \\ s.t. \quad &\hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leqslant \hat{p}_t e_t^t + \hat{p}_{t+1} e_{t+1}^t \\ &c_t^t, \; c_{t+1}^t \geqslant 0 \end{aligned}$$

3. *Markets clear:*

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t, \quad \forall t$$

Sequential Markets Equilibrium

Definition 7 (SME) *A SME is a sequence of interest rates* $\hat{r} = \{\hat{r}_{t+1}\}_{t=1}^{\infty}$*, consumption levels* $\hat{c} = \{\hat{c}_1^0, \{\hat{c}_t^t, \hat{c}_{t+1}^t\}_{t=1}^{\infty}\}$ *and asset holdings* $\hat{b} = \{\hat{b}_{t+1}^t\}_{t=1}^{\infty}$*, such that:*

1. *Given* m*, consumer* 0 *chooses* $\hat{c}_1^0$ *to solve:*

$$\begin{aligned} &\max_{c_1^0} \; u_0(c_1^0) \\ s.t. \quad &c_1^0 \leqslant e_1^0 + m \\ &c_1^0 \geqslant 0 \end{aligned}$$

2. *Given* $\hat{r}_{t+1}$*, each generation* $t \in \{1, \ldots, \infty\}$ *chooses* $\hat{c}_t^t$*,* $\hat{c}_{t+1}^t$ *and* $\hat{b}_{t+1}^t$ *to solve:*

$$\begin{aligned} &\max_{\{c_t^t, c_{t+1}^t, b_{t+1}^t\}} \; u_t(c_t^t, c_{t+1}^t) \\ s.t. \quad &c_t^t + b_{t+1}^t \leqslant e_t^t, \\ &c_{t+1}^t \leqslant e_{t+1}^t + (1 + \hat{r}_{t+1}) b_{t+1}^t, \\ &c_t^t, c_{t+1}^t \geqslant 0 \end{aligned}$$

3. Markets clear:

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t, \quad \forall t \geqslant 1 \tag{Goods}$$

$$\hat{b}_2^1 = m \tag{Bonds}$$

$$\hat{b}_3^2 = m(1 + \hat{r}_2)$$

$$\cdots$$

$$\hat{b}_{t+1}^t = m \prod_{\tau=2}^{t} (1 + \hat{r}_\tau), \quad t \geqslant 2$$

Models with Production

General set-up

1. Firms
   - There are two types of firms, firms producing consumption goods indexed by $j_c = 1, 2, ..., J_c$ and firms producing investment goods indexed by $j_x = 1, 2, ..., J_x$.
   - Consumption good Firms have technology $F_{ct}^{j_c}(k_t^{j_c}, n_t^{j_c})$
   - Investment good Firms have technology $F_{xt}^{j_x}(k_t^{j_x}, n_t^{j_x})$
   - There are two factors of production k and l with prices r_t and w_t at time t.
2. Consumers
   - Consumer i has utility $u_t^i(c_t^i, l_t^i)$ and endowments k_0^i and $\bar{l}_t^i$, $i = 1, \ldots, n$.
   - Consumers are the ones investing x_t^i; the law of motion of capital is as follows $k_{t+1}^i \leqslant x_t^i + (1-\delta)k_t^i$
   - Consumers own the firms, $\theta_{ij_c}^c$ and $\theta_{ij_x}^x$ are consumer i ownership shares of firm j_c and j_x respectively. It follows that $\sum_{i=1}^I \theta_{ij_x}^x = 1, \sum_{i=1}^I \theta_{ij_c}^c = 1, \forall j_x, j_c$.

Arrow-Debreu Equilibrium

An **Arrow-Debreu equilibrium** is prices $\{(p_{ct}, p_{xt}, r_t^k, w_t)\}_{t=0}^\infty$, household's allocations $\{\{c_t^i, x_t^i,\ k_{t+1}^i, n_t^i, l_t^i\}_{t=0}^\infty\}_{i=1}^n$, firm's allocations $\{\{k_{xt}^{j_x}, n_{xt}^{j_x}, x_t^{j_x}\}_{t=0}^\infty\}_{j_x=1}^{J_x}$, and $\{(k_{ct}^{j_c}, n_{ct}^{j_c}, c_t^{j_c}\}_{t=0}^\infty\}_{j_c=1}^{J_c}$, lifetime profits for each household $\{\pi^i\}_{i=1}^n$, such that:

1. Given prices $\{(p_{ct}, p_{xt}, r_t, w_t)\}_{t=0}^\infty$ and lifetime profits, household i, $i = 1, \ldots, n$, choose $\{c_t^i, x_t^i, k_{t+1}^i, n_t^i, l_t^i\}_{t=0}^\infty$, to solve:

$$\max_{\{c_t^i, x_t^i, k_{t+1}^i, n_t^i, l_t^i\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u_t^i(c_t^i, l_t^i)$$

   subject to

$$\sum_{t=0}^\infty (p_{ct}c_t^i + p_{xt}x_t^i) \leqslant \sum_{t=0}^\infty (w_t n_t^i + r_t k_t^i) + \pi^i$$

$$n_t^i + l_t^i \leqslant \bar{l}_t^i\ ,\ \forall t$$

$$k_{t+1}^i \leqslant (1-\delta)k_t^i + x_t^i\ ,\ \forall t$$

$$c_t^i, x_t^i, k_t^i, n_t^i, l_t^i \geqslant 0\ ,\ \forall t$$

$$k_0^i \text{ is given}$$

2. Given prices $\{(p_{xt}, r_t, w_t)\}_{t=0}^\infty$, investment Firm j_x, for $j_x = 1, \ldots, J_x$, chooses $\{(k_t^{j_x}, n_t^{j_x}, x_t^{j_x})\}_{t=0}^\infty$, to solve:

$$\max_{\{(k_t^{j_x}, n_t^{j_x}, x_t^{j_x})\}_{t=0}^\infty} \sum_{t=0}^\infty [p_{xt}x_t^{j_x} - w_t n_t^{j_x} - r_t k_t^{j_x}]$$

subject to

$$x_t^{j_x} \leqslant F_{xt}^{j_x}(k_t^{j_x}, n_t^{j_x})$$

$$x_t^{j_x}, k_t^{j_x}, n_t^{j_x} \geqslant 0 \quad , \ \forall t$$

3. Given prices $\{(p_{ct}, r_t, w_t)\}_{t=0}^{\infty}$, consumption Firm j_c, for $j_c = 1, \ldots, J_c$, chooses $\{(k_t^{j_c}, n_t^{j_c}, c_t^{j_c})\}_{t=0}^{\infty}$, to solve:

$$\max_{\{(k_t^{j_c}, n_t^{j_c}, c_t^{j_c})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} [p_{ct} c_t^{j_c} - w_t n_t^{j_c} - r_t k_t^{j_c}]$$

subject to

$$c_t^{j_c} \leqslant F_{ct}^{j_c}(k_t^{j_c}, n_t^{j_c})$$

$$c_t^{j_c}, k_t^{j_c}, n_t^{j_c} \geqslant 0 \quad , \ \forall t$$

4. Markets Clear:
   (a) Consumption goods market: $\sum_{i=1}^{n} c_t^i = \sum_{j_c=1}^{J_c} c_t^{j_c}$
   (b) Investment goods market: $\sum_{i=1}^{n} x_t^i = \sum_{j_x=1}^{J_x} x_t^{j_x}$
   (c) Labour market: $\sum_{i=1}^{n} n_t^i = \sum_{j_x=1}^{J_x} n_t^{j_x} + \sum_{j_c=1}^{J_c} n_t^{j_c}$
   (d) Capital market: $\sum_{i=1}^{n} k_t^i = \sum_{j_x=1}^{J_x} k_t^{j_x} + \sum_{j_c=1}^{J_c} k_t^{j_c}$

5. Profits are correct:

$$\pi_i = \sum_{j_x=1}^{J_x} \theta_{ij_x}^{x} \Big[\sum_{t=0}^{\infty} (p_{xt} x_t^{j_x} - w_t n_t^{j_x} - r_t k_t^{j_x}) \Big] + \sum_{j_c=1}^{J_c} \theta_{ij_c}^{c} \Big[\sum_{t=0}^{\infty} (p_{ct} c_t^{j_c} - w_t n_t^{j_c} - r_t k_t^{j_c}) \Big] \quad , \ \forall i$$

and

$$\sum_{i=1}^{I} \theta_{ij_x}^{x} = 1, \sum_{i=1}^{I} \theta_{ij_c}^{c} = 1, \forall j_x \in J_x, j_c \in J_c$$

Simplification

- A.1 Let $F_{ct}^{j_c}$ and $F_{xt}^{j_x}$ be CRS. As a result $\pi^i = 0$.
- A.2 Let $F_{ct}^{s} = F_{xt}^{t} = F$, $\forall s, t$, the same firm produces both. As the goods are substitutes in production they have the same price.
- A.3.1 Assume representative agent: $\beta^i = \beta$, $u^i = u$, $k_0^i = k_0$ and $\bar{l}_t^i = \bar{l}_t$, $\forall i, t$.
- A.3.2 Homothetic aggregation: $\beta^i = \beta$, $u^i = u$, $\forall i$ and u is homothetic. We do $\bar{l}_t = \sum_i \bar{l}_t^i$ and $k_0 = \sum_i k_0^i$. Solving the maximization problem for the aggregate we can then retrieve each agent's choices: $c_t^i = \alpha^i c_t$ and $x_t^i = \alpha^i x_t$ where:

$$\alpha^i = \frac{r_0 k_0^i + p_0(1-\delta) k_0^i + \sum_{t=0}^{\infty} w_t \bar{l}_t^i}{\sum_i \left(r_0 k_0^i + p_0(1-\delta) k_0^i + \sum_{t=0}^{\infty} w_t \bar{l}_t^i \right)}$$

Simplified Arrow-Debreu Equilibrium

An ADE is a sequence of prices $P = \{\hat{p}_t, \hat{w}_t, \hat{r}_t\}_{t=0}^{\infty}$, allocation for the consumer $Z^H = \{\hat{c}_t, \hat{n}_t, \hat{l}_t, \hat{k}_{t+1}\}_{t=0}^{\infty}$ and allocation for the firm $Z^F = \{\hat{y}_t^f, \hat{n}_t^f, \hat{k}_t^f\}_{t=0}^{\infty}$, such that:

1. Given P, consumer chooses Z^H to solve:

$$\begin{aligned} &\max_{Z^H} \sum_{t=0}^{\infty} \beta^t u(c_t, \bar{l}_t - n_t) \\ \text{s.t.} \quad &\sum_{t=0}^{\infty} \hat{p}_t[c_t + k_{t+1} - (1-\delta)k_t] \leqslant \sum_{t=0}^{\infty} [\hat{w}_t n_t + \hat{r}_t k_t] \\ &c_t, k_{t+1} \geqslant 0 \quad \forall t \\ &k_0 \text{ given} \end{aligned}$$

2. Given P firms minimize cost and earn zero profit due to free entry:

$$\sum_{t=0}^{\infty} (\hat{p}_t \hat{y}_t^f - \hat{w}\hat{n}_t^f - \hat{r}_t \hat{k}_t^f) \leqslant 0, = 0 \text{ if } \hat{y}_t^f > 0$$

   Where $\hat{y}_t^f = F(\hat{k}_t^f, \hat{n}_t^f)$. Write the factor prices equal the marginal products (firm's first order conditions).

   Or (less preferred):

   Given P, firm chooses Z^F to solve:

$$\begin{aligned} &\max_{Z^F} \sum_{t=0}^{\infty} [\hat{p}_t y_t^f - \hat{w}_t n_t^f - \hat{r}_t k_t^f] \\ \text{s.t.} \quad &y_t^f \leqslant F(k_t^f, n_t^f) \\ &n_t^f, k_t^f \geqslant 0 \quad \forall t \end{aligned}$$

3. Markets clear:

$$\hat{c}_t + \hat{k}_{t+1} = F(\hat{k}_t, \hat{n}_t) + (1-\delta)\hat{k}_t, \quad \forall t \tag{Goods}$$

$$\hat{k}_t^f = \hat{k}_t, \quad \forall t \tag{Capital}$$

$$\hat{n}_t^f = \hat{n}_t, \quad \forall t \tag{Labor}$$

Overlapping Generations Model (with exogenous labor)

Definition 8 (ADE) *An ADE is a sequence of prices* $\hat{p} = \{\hat{p}_t, \hat{w}_t, \hat{r}_t\}_{t=1}^{\infty}$*, allocations for the consumer* $Z^H = \{\hat{c}_1^0, \{\hat{c}_t^t,\ \hat{c}_{t+1}^t, \hat{k}_{t+1}^t\}_{t=1}^{\infty}\}$ *and allocations for the firm* $Z^F = \{\hat{l}_t^f, \hat{k}_t^f, \hat{y}_t^f\}_{t=1}^{\infty}$*, such that:*

1. *Given* $\hat{p}_1, \hat{w}_1, \hat{r}_1$ *and* m*, generation* 0 *chooses* $\hat{c}_1^0$ *to solve:*

$$\begin{aligned} &\max_{c_1^0}\ u_0(c_1^0) \\ s.t.\quad &\hat{p}_1 c_1^0 \leqslant w_1 \bar{l}_1^0 + (\hat{p}_1(1-\delta) + \hat{r}_1)k_1^0 + m \\ &c_1^0 \geqslant 0 \end{aligned}$$

2. *Given* $\hat{p}_t,\ \hat{p}_{t+1},\ \hat{w}_t, \hat{w}_{t+1}$ *and* $\hat{r}_{t+1}$ *generation* t *chooses* $\hat{c}_t^t,\ \hat{c}_{t+1}^t$ *and* $\hat{k}_{t+1}^t$ *to solve, for all* $t \in \{1, \ldots, \infty\}$*:*

$$\begin{aligned} &\max_{\{c_t^t, c_{t+1}^t, k_{t+1}^t\}}\ u_t(c_t^t, c_{t+1}^t) \\ s.t.\quad &\hat{p}_t(c_t^t + k_{t+1}^t) + \hat{p}_{t+1}c_{t+1}^t \leqslant \hat{w}_t \bar{l}_t^t + \bar{w}_{t+1}\bar{l}_{t+1}^t + (\hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1})k_{t+1}^t \\ &c_t^t,\ c_{t+1}^t, k_{t+1}^t \geqslant 0 \end{aligned}$$

3. *Given* P *firms minimize cost and earn zero profit due to free entry:*

$$\sum_{t=0}^{\infty}(\hat{p}_t\hat{y}_t^f - \hat{w}\hat{l}_t^f - \hat{r}_t\hat{k}_t^f) \leqslant 0, = 0 \text{ if } \hat{y}_t^f > 0$$

   Where $\hat{y}_t^f = F(\hat{k}_t^f, \hat{n}_t^f)$*. Write the factor prices equal the marginal products (firm's first order conditions).*

   Or (less preferred):

   Given $\hat{w}_t$ *and* $\hat{r}_t$ *firms choose* $\hat{l}_t^f, \hat{k}_t^f, \hat{y}_t^f$*,* $\forall t \geqslant 1$*, to solve:*

$$\begin{aligned} &\max_{Z^F}\ \hat{p}_t y_t^f - \hat{w}_t l_t^f - \hat{r}_t k_t^f \\ s.t.\quad &y_t^f \leqslant F(k_t^f, n_t^f) \\ &l_t^f, k_t^f \geqslant 0 \end{aligned}$$

4. *Markets clear:*

$$\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} = F(\hat{k}_t, \bar{l}_t^t + \bar{l}_t^{t-1}), \quad \forall t$$

$$\hat{l}_t^f = \bar{l}_t^t + \bar{l}_t^{t-1}$$

$$\hat{k}_t^f = \hat{k}_t^{t-1}$$

Dynamic Programming

Please read chapters 3 and 4 of SLP

Social Planner's Problem

$$\begin{aligned} V(k_0) = & \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c_t \\ \text{s.t.} \quad & c_t + k_{t+1} \leqslant A k_t^{\alpha} \\ & c_t, \ k_{t+1} \geqslant 0 \\ & k_0 \ \text{ given} \end{aligned}$$

Set-up: log utility, Cobb-Douglas production function (inelastic labor); as a result we have an analytical solution.

$$\begin{aligned} V(k_0) = & \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(A k_t^{\alpha} - k_{t+1}) \\ \text{s.t.} \quad & 0 \leqslant k_{t+1} \leqslant A k_t^{\alpha} \\ & k_0 \ \text{ given} \end{aligned}$$

Euler equation as a second order difference equation.

$$-\frac{1}{A k_t^{\alpha} - k_{t+1}} + \beta \frac{A \alpha k_{t+1}^{\alpha-1}}{A k_{t+1}^{\alpha} - k_{t+2}} = 0$$

Transversality Condition:

$$\lim_{t \to \infty} \frac{\beta^t}{A k_t^{\alpha} - k_{t+1}} k_{t+1} = 0$$

Recursive Statement:

$$\begin{aligned} V(k) = & \max_{k'} \quad \log(A k^{\alpha} - k') + \beta V(k') \\ \text{s.t.} \quad & 0 \leqslant k' \leqslant A k^{\alpha} \end{aligned}$$

The result we are looking for: $k' = g_k(k)$ and $c = g_c(k)$, i.e. a first order difference equation - the current state (k) gives you the optimal k' and c.

We will follow two different approaches, Guess and Verify and Value Function Iteration.

Guess and Verify

Let us guess the following:

$$V(k) = a_0 + a_1 \log k$$

Then we get the following first order condition and envelope condition:

$$-\frac{1}{Ak^\alpha - k'} + \beta V'(k') = 0 \iff \frac{1}{Ak^\alpha - k'} = \beta \frac{a_1}{k'}$$

$$V'(k) = \frac{\alpha A k^{\alpha-1}}{Ak^\alpha - k'} \iff \frac{a_1}{k} = \frac{\alpha A k^{\alpha-1}}{Ak^\alpha - k'}$$

For any given k we have two equations and two unknowns.

$$k' = \frac{\beta a_1}{1+\beta a_1} A k^\alpha$$

$$a_1 = \frac{\alpha A k^\alpha}{Ak^\alpha - k'} = \frac{\alpha A k^\alpha}{Ak^\alpha - \frac{\beta a_1}{1+\beta a_1} A k^\alpha} = \frac{\alpha}{1 - \frac{\beta a_1}{1+\beta a_1}} \iff a_1 = \frac{\alpha}{1-\alpha\beta}$$

Then we get:

$$g_k(k) = \alpha\beta A k^\alpha$$

$$g_c(k) = (1-\alpha\beta) A k^\alpha$$

To get $V(k)$ we need to find a_0 and a_1. To do that we can plug in $k'(a_1)$ into the original problem:

$$V(k) = \log\left(Ak^\alpha - \frac{\beta a_1}{1+\beta a_1} A k^\alpha\right) + \beta V\left(\frac{\beta a_1}{1+\beta a_1} A k^\alpha\right)$$

Then, notice that:

$$LHS: \; a_0 + a_1 \log k$$

$$RHS: \; C + D \log k$$

where, C and D are some constants.

Then, $a_0 = C$ and $a_1 = D = \frac{\alpha}{1-\alpha\beta}$.

Now we need to **Verify** the guess. We verify the Euler equation and the transversality condition.

$$\begin{aligned}
-\frac{1}{Ak_t^\alpha - k_{t+1}} + \beta \frac{A\alpha k_{t+1}^{\alpha-1}}{Ak_{t+1}^\alpha - k_{t+2}} &= \\
&= -\frac{1}{1-\alpha\beta}\frac{1}{Ak^\alpha} + \alpha\beta \frac{Ak_{t+1}^{\alpha-1}}{(1-\alpha\beta)} \frac{1}{Ak_{t+1}^\alpha} \\
&= -\frac{1}{1-\alpha\beta}\frac{1}{Ak^\alpha} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k_{t+1}} \\
&= -\frac{1}{1-\alpha\beta}\frac{1}{Ak^\alpha} + \frac{1}{1-\alpha\beta}\frac{1}{Ak_t^\alpha} \\
&= 0
\end{aligned}$$

$$\lim_{t\to\infty} \frac{\beta^t}{Ak_t^\alpha - k_{t+1}} k_{t+1} = \lim_{t\to\infty} \frac{\beta^t}{(1-\alpha\beta)Ak_t^\alpha} \alpha\beta A k_t^\alpha = \frac{\alpha\beta}{1-\alpha\beta} \lim_{t\to\infty} \beta^t = 0$$

Value Function Iteration

1. Start with a guess, for simplicity $V_0(k) = 0$;
2. Solve $V_1(k)$:

$$V_1(k) = \max_{k'} \quad \log(Ak^\alpha - k') + \beta V_0(k')$$
$$\text{s.t.} \quad 0 \leqslant k' \leqslant Ak^\alpha$$

   The first order condition yields:

$$-\frac{1}{Ak^\alpha - k'} + \gamma k' = 0$$
$$\gamma k' = 0 \quad \gamma \geqslant 0 \quad k' \geqslant 0$$

   The solution is simply $g_1(k) = 0$ and as a result:

$$V_1(k) = \log Ak^\alpha$$

3. Solve $V_2(k)$:

$$V_2(k) = \max_{k'} \quad \log(Ak^\alpha - k') + \beta V_1(k')$$
$$\text{s.t.} \quad 0 \leqslant k' \leqslant Ak^\alpha$$

   The first order condition yields:

$$-\frac{1}{Ak^\alpha - k'} + \beta \frac{\alpha A(k')^{\alpha-1}}{A(k')^\alpha} = 0$$

   We get the following:

$$g_2(k) = \frac{\alpha\beta}{1+\alpha\beta} Ak^\alpha$$
$$V_2(k) = \log\left(\frac{1}{1+\alpha\beta} Ak^\alpha\right) + \beta \log\left(A\left[\frac{\alpha\beta}{1+\alpha\beta} Ak^\alpha\right]^\alpha\right) = \alpha(1+\alpha\beta)\log k + \textit{constant}$$

4. Repeat the steps.

By the contraction mapping theorem you know that the iteration will be converging to V.

Theory

Why are we confident that Value Function Iteration gives us V^*?

In general we have:

- An initial guess $V_0(k) = 0$;
- $V_{n+1}(k) = \max_{k'} u(F(k) + (1-\delta)k - k') + \beta V_n(k') \iff V_{n+1}(k) = T(V_n)(k)$

We want:

$$\lim_{n\to\infty} V_{n+1}(k) = V^*(k), \quad \text{where} \quad V^*(k) = T(V^*)(k)$$

Let $X \subseteq \mathbb{R}^l$, let $B(X)$ be the set of bounded functions $f : X \to \mathbb{R}$.

Let $d(V,W) = ||V(k) - W(k)|| = \sup_{k\in K} |V(k) - W(k)|$ be the sup norm.

Definition 9 (Cauchy Sequence) *A sequence $V_0, V_1, \ldots, V_n \in B$ is a Cauchy sequence if $\forall \epsilon > 0$, there exists N_ϵ such that:*

$$d(V_n, V_m) < \epsilon \quad \forall n, m > N_\epsilon$$

Definition 10 (Convergence) *A sequence $V_0, V_1, \ldots, V_n \in B$ converges to $V^* \in B$ if $\forall \epsilon > 0$, there exists N_ϵ such that:*

$$d(V_n, V^*) < \epsilon \quad \forall n > N_\epsilon$$

Definition 11 (Complete metric space) *A metric space (S, d) is complete if every Cauchy sequence in S converges to an element in S.*

The set of bounded functions B with the sup norm is a complete normed vector space.

Definition 12 (Contraction Mapping) *Let $(B(X), d)$ be a vector space. $T : B(X) \to B(X)$ is a contraction mapping with modulus β if for some $\beta \in (0,1)$, we have*

$$d(TV, TW) \leqslant \beta d(V,W) \quad \forall V, W \in B(X)$$

Theorem 4 (Contraction Mapping Theorem) *If $(B(X), d)$ is a complete metric space and $T : B(X) \to B(X)$ is a contraction mapping with modulus β, then:*

1. *T has exactly one fixed point V^* in $B(X)$, and*
2. *for any $V_0 \in B(X)$, $d(T^n(V_0), V^*) \leqslant \beta^n d(V_0, V^*)$ for $n = 0, 1, 2, \ldots$*

We are saying that if f is any function in $B(X)$ and $\{f_n\}$ a sequence that is defined inductively $f_{n+1} = T(f_n)$, then $\{f_n\}$ converges to a unique fixed point.

Theorem 5 (SLP Theorem 3.3 - Blackwell's Sufficient Conditions) *Let $X \subseteq \mathbb{R}^l$, let $B(X)$ be the space of bounded functions $f : X \to \mathbb{R}$ with the sup-norm. Let $T : B(X) \to B(X)$ be an operator satisfying:*

1. *Monotonicity*

   Let $f, g \in B(X)$ and $f(x) \leqslant g(x)$, $\forall x \in X$, then $(Tf)(x) \leqslant (Tg)(x)$, $\forall x \in X$.

2. *Discounting*

There exist $\beta \in (0,1)$ such that $[T(f+a)](x) \leqslant (Tf)(x) + \beta a$, $\forall f \in B(X)$, $a \geqslant 0$, $x \in X$, where $(f+a)(x) = f(x) + a$

Then T is a contraction with modulus β.

Let's get back to our baby growth model example:

$$T(V)(k) = \max_{0 \leqslant k' \leqslant Ak^\alpha} \log(Ak^\alpha - k') + \beta V(k')$$

$X = [0, \tilde{K}]$ where $\tilde{k}$ is the maximum sustainable level of capital. Note that if k is bounded then the utility is also bounded above.

Let $V, W : X \to \mathbb{R}$ be bounded such that $V(x) \leqslant W(x)$, $\forall x \in X$, then we have:

$$\log(Ak^\alpha - k') + \beta V(k') \leqslant \log(Ak^\alpha - k') + \beta W(k'), \quad \forall k' \in [0, Ak^\alpha]$$
$$\max_{0 \leqslant k' \leqslant Ak^\alpha} \log(Ak^\alpha - k') + \beta V(k') \leqslant \max_{0 \leqslant k' \leqslant Ak^\alpha} \log(Ak^\alpha - k') + \beta W(k')$$
$$T(V)(k) \leqslant T(W)(k)$$

Hence, monotonicity is checked. □

$$\begin{aligned} T(V+a)(k) &= \max_{0 \leqslant k' \leqslant Ak^\alpha} \log(Ak^\alpha - k') + \beta(V(k') + a) \\ &= \max_{0 \leqslant k' \leqslant Ak^\alpha} \log(Ak^\alpha - k') + \beta V(k') + \beta a \\ &= T(V)(k) + \beta a \end{aligned}$$

Also, $\beta \in (0,1)$ by assumption. Hence, discounting is also checked. □

T is a contraction with modulus β and we can apply the Contraction mapping theorem - there is convergence - one unique fixed point.

Maximum sustainable capital

The maximum sustainable capital stock $\tilde{K} = \max\{k_0, \bar{k}\}$ where $\bar{k}$ is such that

$$\bar{k} - (1-\delta)\bar{k} = A\bar{k}^\alpha \iff \bar{k} = \left(\frac{A}{\delta}\right)^{\frac{1}{1-\alpha}}$$

You save everything and do not consume.

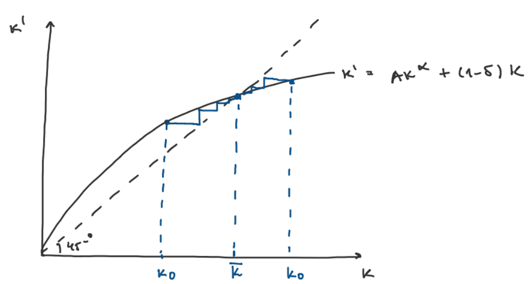

Figure 1: Maximum sustainable capital stock

Pseudo-code (VFI for baby growth model)

1. Set values for parameters: $\alpha, \beta, \delta, \theta$
2. Create grid for capital, a vector with the possible elements that you want k and k' to take. In Julia you can use the command: $k = range(kmin, stop = kmax, length = nbk)$
3. Create an array for consumption. If you have 200 possible values for capital then the consumption matrix is (200×200). You can do this with a for loop: for each value of k and k' in the grid compute consumption using the resources constraint. It is a good idea to create an empty array first and then fill it with the results you obtained. Also note that if $\delta < 1$ you need to rule out cases with negative invstment.
4. Compute utility from consumption. Note that you need to rule out negative consumption.
5. Set auxiliary variables/parameters for VFI:
   - Define a tolerance and an initial value for the sup norm;
   - Set a counter equal to zero for the number of iterations and set a maximum number of iterations;
   - Set two arrays, one for the value function at the beginning of iteration and one to store the value function after the iteration, both full of zeros.
6. Proceed to Value Function Iteration:

   Do while $error > tol$ and until your iteration is smaller than the limit you have set

   (a) increase counter of iterations by 1;

   (b) for each value of k maximize over k_1, a useful command in Julia is:

   $$V_{new}[k], k_{index}[k] = findmax(u[k,:]. + \beta * V_{old}[:])$$

   (c) Compute the supnorm as the maximum of the absolute value of the difference between V_{new} and V_{old};

   (d) Copy V_{new} to V_{old}.
7. Get the policy function for capital and consumption: note that $k_{index}[k]$ gives you the position in the grid of k that maximizes utility, hence, to get the policy functions we do:

   (a) Capital: do a for loop over k, for each value of k see the value corresponding to the element in the grid that is the maximizer;

   (b) Consumption: do a for loop over k, for each value of k compute consumption using the Resources constraint and taking into account the policy function for capital.
8. Plot your results.